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Time and Timelessness in Fundamental Physics and Cosmology

Historical, Philosophical, and Mathematical Perspectives

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Foreword

On April 8, 1610, Johannes Kepler received Galileo Galilei's epoch-making book *Sidereus Nuncius*. With great enthusiasm, he wrote a detailed response, which he could send to Galileo already on April 19: the *Dissertatio cum Nuncio Sidereo* (Conversation with the Starry Messenger). In a sense, this can be considered the beginning of physical cosmology and has also an indirect bearing on the issue of time. In his response, Kepler formulated what is today known as Olbers' paradox: if the universe is homogeneously filled with stars (as Galileo's discovery of hitherto unseen stars seems to indicate), the night sky should, in fact, look brighter than the Sun. Kepler concluded from this that the universe must be finite. From the law of universal gravity, Newton later argued that the universe must be infinite in size, for in a finite universe there would be a center of gravity, so that everything would "fall down into the middle of the whole space, and there comprise one great spherical mass." This collapse would have happened in a finite *time*, already over in an eternal universe. This is in contradiction to observation, so the universe must be infinite.

The relation between the universe as a whole and the concept of (cosmic) time continues to play a central role in modern cosmology. It is also the topic of this volume which brings together a careful selection of contributions addressing these issues from an interdisciplinary perspective, combining physics, astronomy, mathematics, and philosophy.

Modern cosmology is based on Einstein's theory of general relativity (GR). In it, time is no longer absolute, but part of a dynamical spacetime. In the decades after the completion of the theory in 1915, there were only a few reliable cosmological observations, among them the discovery of cosmic expansion and of the Cosmic

Microwave Background (CMB) as a relic from an early radiation-dominated phase of the universe. It is the last decade of the twentieth century and the twenty-first century that saw the advent of *precision cosmology*. The satellite PLANCK with its successful measurements of the CMB anisotropy spectrum (describing the seeds of cosmic structure formation) may serve as one example. We now know that the expansion of the universe is currently accelerating and we know its age with unprecedented accuracy: about 13.8 billion years. Many terrestrial and space missions are in preparation, in particular, devoted to the observation of gravitational waves, which could open a new window to the early universe, much earlier than the release of the CMB. This could give final support to the idea of the inflationary universe, which describes a cosmic acceleration at a very early state, providing the means of solving various cosmic riddles and the key to structure formation.

So far, Einstein's theory is not in conflict with observations, although it is often claimed that two phenomena—described by the words Dark Matter and Dark Energy—point toward its generalization. This may be the case, but it is far from clear. Dark Matter may consist of unknown particles (such as axions) that interact gravitationally in the usual way, and Dark Energy, responsible for the accelerated expansion of our universe, may in fact be Einstein's cosmological constant. But is, on the theoretical side, Einstein's theory of GR the last word? This is very unlikely. All non-gravitational interactions so far are successfully described by quantum (field) theories, culminating at present in the Standard Model of particle physics. Given the universal coupling of gravity to all forms of energy, it seems unlikely that gravity is not described by a quantum theory at the most fundamental level. Moreover, the singularity theorems of GR indicate the incompleteness

of the theory. For this reason, most scientists believe that GR should be superseded by a quantum theory of gravity. Such a theory is still elusive, but various approaches exist. It is clear that such a theory has an important bearing on cosmology as is reflected in the title of this volume.

The most conservative approach to quantum gravity is the direct quantization of GR, which goes back to Léon Rosenfeld's pioneering work of 1930. After important work by Paul Dirac, Peter Bergmann and his group, and Arnowitt, Deser, and Misner (ADM), John Wheeler and Bryce DeWitt were able to present in the 1960s for the first time a concrete quantum version of GR beyond the perturbative level. Time, and therefore spacetime, is completely absent in the fundamental equations of quantum GR. At first glance, this may look surprising, but it becomes obvious at second thought. In the traditional way of quantization, spacetime is analogous to a particle trajectory in mechanics. Since such trajectories are no longer present in quantum mechanics, being replaced by wave functions, spacetime is absent in quantum gravity, and only space remains. The importance of this issue is recognized from the title of this volume, and many of its contributions address it in a conservative or a radical way.

So I invite the reader to delve into this fascinating field of time, cosmology, and fundamental physics, and follow the thought-provoking contributions collected in this volume. As Newton once remarked: "I keep the subject constantly before me, and wait till the first dawnings open slowly, by little and little, into a full and clear light."

Claus Kiefer
Cologne, Germany
March 2024

Introduction

This volume is the result of a long-lasting reflection upon the central question involving the emergence or disappearance of time discussed in fundamental physics and cosmology. Current physics has to face challenges related to the assessment of the status of spacetime within and beyond relativity. “Is spacetime fundamental?” If the answer to this question is negative, then we enter the realm of new frontiers in theoretical physics, which include among the proposals the radical revision of GR or the formulation of new theories, including quantum gravity and string theory. This move in turn raises several questions regarding the nature of time and timelessness in models of quantum gravity and quantum cosmology. For instance, one can ask questions regarding the conditions under which an emergence of spacetime occurs and inquire about the nature of “emergence” in this context.

If spacetime is emergent from quantum physics it is clear that a theory of quantum gravity is also expected to recover its dimensionality and metric to some approximation and, at the same time, it should describe a pre-geometric realm in which time could play no role or a very different one from relativistic physics. In recent times, Carlo Rovelli, among others, had the merit of having clarified in several ways the relational nature of time already implied by relativity, thereby corroborating the idea that any theory of quantum gravity must match or be compatible with this tenet. Loop Quantum Gravity (LQG), for instance, suggests the “disappearance” of classical time at the fundamental scale and provides the description of the covariant dynamics of fundamental constituents of spacetime. However, the domain of quantum gravity seems to be a zoo populated by several proposals that treat emergent time in different ways, for example: Causal

Dynamical Triangulation (CDT), Causal Set Theory, or Group Field Theory (GFT).

But what is time? There are several definitions of time in quantum physics and cosmology. Leaving aside the definition of proper time and cosmological time in the relativistic framework, we are accustomed to talk about internal time in Quantum Cosmology when considering the Wheeler–DeWitt equation, or we talk about the time when we associate a clock to a scalar field (e.g., volume) in GFT cosmology. Shouldn't we rather connect it to matter fields? A discourse involving time in Euclidean path integral approaches to quantum gravity even involves Euclidean time or imaginary time when modeling the universe or black holes. Thus, it seems that the polysemous nature of time complicates the problem, since no univocal definition of time at the fundamental scale seems possible to attain, at least for the time being. Is time simply related to a parameter measuring intervals? Is it just a clock or something serving as a clock through which we describe evolution of a system for different observers? Is time more fundamental than a pure label and capable of disclosing the physics of exotic scales and objects such as black holes? These questions are addressed in this volume from both philosophical and physical perspectives and in a range of different approaches to quantum gravity and cosmology.

The novelty of this volume lies in the interaction among scientists, philosophers, and historians in exploring the nature of time and timelessness and the origin of these concepts. Indeed, the problem is one of the oldest, if not the oldest one, in Western culture. It can be traced back to Plato's *Timaeus*, an extraordinary masterpiece, a dialogue in which the origin of the world and the nature of time are core questions together with the reflection on the fundamental geometrical constituents of matter. The dialogue also presents a discussion of the proto-geometric

structure of the world, which makes it one of the most relevant sources for a preliminary philosophical reflection upon timelessness and proto-temporality in physics. For this reason, we decided to structure the volume in three parts. The first one is devoted to the historical discussion of time and timelessness from the Platonic tradition to German Idealism, and emphasizing its impact throughout the centuries, up to Gödel. The second part of this volume presents five contributions that aim at discussing the role of time and its emergence in fundamental physics and causal set theory. More specifically, the authors explore the fundamentality of time in general relativity, the role of causal relations, and the emergence of continuous structures from discrete entities. This in turn allows the reader to obtain an enriched picture of the problems that physics is urged to address and at the same time stimulates further reflections that can perhaps avoid the repetition of old patterns already devised in the history of science and philosophy of nature. Indeed, one of the most important messages that the book wants to convey is the fact that from Aristotle onward we conflated the notion of timelessness with eternity and this in turn limited the way in which philosophers and physicists could think of time and temporal series. To become aware of this limitation, one needs to go back to Plato and his dialogues and appreciate the complex nature of his theory of time and atemporality. Once this is revealed, one can identify the conceptual limitations that current physics and cosmology carry with them and think about suitable alternative scenarios, thereby stimulating a novel understanding of time and timelessness, which is not a small objective today.

Part I of the volume offers an overview that reconstructs important aspects of the debate over the timeless account of the beginning of the world within the Platonic tradition and some of its reappraisals in modern times. This in turn sheds light on the affinities and differences between

ancient and modern debates on the philosophy of time in cosmology. The first contribution by Laura Marongiu, entitled “Atemporality and the Origins of the Eternal Cosmos: Debates on Timeless Simultaneity Within Platonic Cosmogonies,” explores the notion of atemporality within selected works characterized by cosmogonical views denying that the cosmos is generated in time. A specific notion of atemporality, namely simultaneity or “at once” (ἀθρόως), is investigated by authors such as Plotinus, Porphyry, Calcidius, and Proclus. This examination sheds light on how these Platonists, by appealing to a multi-layered taxonomy of atemporality, can consistently portray the cosmos as being both ungenerated and “created” all at once.

The second contribution “Time and Atemporality of Time in Hegel’s Naturphilosophie” by Laura Follesa focuses on the relations between philosophy and the sciences during the early-nineteenth-century philosophy and offers a detailed analysis of the work of the German philosopher Georg Friedrich Hegel. In recent times, readings of his writings have revealed a genuine attempt to reconcile metaphysics and the sciences. This attempt becomes evident when examining Hegel’s treatment of the problem of time, especially in its relationship with the notions of “world” and of “life” and opens the path for a reflection over the concept of atemporality in Hegel’s philosophy of nature. Follesa shows the direct impact of Plato’s *Timaeus* and Kant on Hegel’s view of cosmology and underlines the connection that Hegel suggests between the problem of the beginning of the universe in time and the perspective from which we represent the idea of time. According to an ordinary way of thinking, the world is only an aggregate of finite existences, but when it is grasped as a whole the question of a beginning at once disappears. There is no definite place in which the beginning can happen or can be

located through a coordinate system and this means that any scientific model can only identify a relative beginning of the world.

In “Not Even Ideal. Kant’s Critique of Absolute Time and Gödel’s Rotating Universes,” Silvia De Bianchi shows how the impact of Kant’s philosophy shaped the ideas about time (and space) that stimulated Kurt Gödel in finding a particular set of solutions of Einstein Field Equations in between 1949 and 1952. The rotating universe model has been largely inspired by Gödel’s reading of Kant’s *Metaphysical Foundations of Natural Science* (1786) in which Kant provided an argument for the relativity of motion of reference frames and for the rejection of absolute time in cosmology. What emerges from the discussion is an asymmetric treatment of space and time in cosmology, resulting in the fact that for both Kant and Gödel absolute time is not even ideal and must be removed from cosmological models. What is rather suggested is the emergence of a local world time and this of course creates the condition for a cosmological account that represents the world as a timeless whole in which only dynamical local time can be admitted.

Part II of the volume introduces the discussion of contemporary modern physics and contains contributions that discuss the notion of time in theories of quantum gravity and, more specifically, in causal set theory. The first contribution, “Time is Order,” by Álvaro Mozota-Frauca, challenges the view that many approaches to quantum gravity seem to adopt with respect to the fundamentality and emergence of time. First, the contribution comments on the twofold role of time in classical theories in terms of (i) order relation between events and (ii) definition of temporal intervals. The distinction aims at showing that recent attempts to characterize physical quantities as correlations between physical observables, in some

quantum gravity approaches, leave out important physical information. On the one hand, canonical approaches to quantum gravity seem to derive a trivial evolution equation in which time plays no role. Relationists argue that this should not be considered much of a problem, for one can build some appropriate quantum operators that capture a correlation between two observables. On the other hand, others have suggested adopting a functionalist view of spacetime for which one would need to recover the functions played by spacetime, even if instantiated by non-spatiotemporal entities. In the former case, the author insists that the formulation of correlation operators between observable quantities might leave out important physical information encoded in the order of events. In the latter case, the author concludes that if the main function of spacetime is to be recovered to support the functionalist account, then that function should be the time ordering between events.

The second contribution to Part II, “En Route to Reduction: Lorentzian Manifolds and Causal Sets,” by Jeremy Butterfield, discusses the causal set program as a case study for the philosophical contrast between emergence, reduction, and for spacetime functionalism. With respect to the former, Butterfield maintains that, while the two notions are typically characterized as contraries, more recent philosophical literature has advocated for their compatibility. Such a compatibility, also defended by the author, is based on the idea that some mathematical structures emerge at the limit of certain parameters and that the theory ensuing from them should be considered as genuinely emergent. Yet, the emergent theory is still reducible “before the limit,” therefrom the compatibility between reduction and emergence. What matters, both scientifically and philosophically, is the theory *before the limit*, that is, when the emergent

properties are only foreshadowed, or the emergent behavior is seen *en route.* Then, with respect to functionalist reduction, the author points out how we do not have a proper and precise formulation of the functional roles of the concepts of interest, and, even if we did, they might not be fulfilled at the bottom level, due to the differences between discrete and continuous spaces. Nonetheless, the contribution suggests a roadmap for achieving functionalist reduction. First (i), one needs to show that the concepts of interest such as dimension, spacelike length, and curvature are specified by their functional role. Second (ii), one needs to show that the analogous functional roles are sufficiently analogous to (i) so as to give a functionalist reduction. Again, the conclusion is similar to that of Nagelian reduction: there is no in-principle incompatibility of causal set theory with functionalist reduction, but quite some work is still needed.

In continuity with the previous contributions, Marco Forgione in “Causation as Constraints in Causal Set Theory” focuses on the role of time in quantum gravity, and on its definition in terms of ordering between events—which Mozota-Frauca characterized as one of the two roles of time in physics. Like Butterfield, Forgione sets on discussing the role of time in a specific approach to quantum gravity: causal set theory. The contribution contends that the causal relation of causal set theory can be characterized as an atemporal constraint in the theory’s kinematic space. This relation, which is formally defined as a partial order, places restrictions on the birth of a new element (or event) with respect to the other elements or events within the same causal set. Consequently, the progression of time emerges from the various potential growths of the specified set, each possibility being distinguished by a classical probability determined by the dynamics of the theory. Thus, the author maintains that

causation, in the form of partial ordering, is ontologically prior to time, and that the emergence of time is also due to the constraints imposed by the causal relation on the possible configurations of the causal set in the kinematic space. The second part of the contribution considers the non-identity approach, as articulated by (Baron and Le Bihan 2023), which establishes a grounding relationship between spacetime and causal relations within causal set theory. Forgione reaches the same conclusion, though he dispenses with the ontological relation in favor of deriving the priority of causal relations over spacetime from the axioms of the theory. Finally, the chapter explores some accounts of causation to characterize the partial order of causal set theory and concludes that causal relations act as a super-selection rule over the space of possible growths. In this sense, causal relations are atemporal constraints over the possible causal sets, and the collective action of such constraints, together with a yet unspecified mechanism of quantum interference between causal paths, should lead to the emergence of spacetime and the phenomenological flow of time.

In Chap. 7, titled “Canonical Covariant Quantization and the Problem of Time,” Steven Carlip and Weixuan Hu discuss the problem of time in quantum gravity in view of the covariant phase space picture. More specifically, they suggest that a quantized version of the classical covariant phase space view might be of help to solve the problem of time in quantum gravity. The contribution begins with three evident aspects of the problem of time in quantum gravity. (i) The Wheeler De-Witt equation has no temporal derivative, which seems to suggest that physical states are independent of time. (ii) Quantum mechanics requires an inner product operation between states “at the same time,” but the time coordinate has vanished from the Wheeler De-Witt equation. Finally (iii), the causal structure of spacetime in quantum field theory obeys the condition of

micro-causality, for which two operators acting on two spacelike separated events commute. This condition is problematic in quantum gravity because whether two states are spacelike separated or not depends on the time slicing. The solution explored in the chapter is to consider the covariant phase space formulation of classical mechanics and to attempt its quantization.

In covariant phase space, each point corresponds to an entire classical history since, for a theory with well-posed initial values problem, each initial set of values will evolve in a way that defines a unique classical solution: a history. In other words, if we have well-defined initial values, the dynamics law will evolve the initial values deterministically and uniquely. It follows that it is enough to have the initial values of the system to have the system's entire history. The idea is to reformulate phase space as the space of those histories. Carlip and Hu prove that the phase space of initial data and the phase space of classical histories are mathematically isomorphic, but physically very different. The ordinary phase space implies that some initial data at fixed time are evolved in their future counterparts. On the other hand, the covariant picture is time-independent since the time of the histories is a choice of coordinates. Upon quantization, covariant phase space gives time-independent wave functions which can be interpreted as amplitudes for block universes. But then, the authors obtain a set of slice-dependent observables that will become time-dependent operators in the quantum theory. The initial time on any initial time slice can play the role of coordinates for the covariant space, where the time evolution acts as coordinates change. When quantized, the change of coordinates becomes a change of basis that depends on the initial time slicing, and each time slicing corresponds to a different description of the universe. Finally, the authors provide an example of the new framework in (2+1)-dimensional gravity in which they emphasize the limits and

difficulties of the new approach. If those limits and difficulties turn out to be “only technical,” then hard work might be enough to overcome some of the conceptual problems of quantum gravity.

In Chap. 8, titled “Asymptotic Reasoning and Universality in (Space) Time Dynamics,” José Luis Jaramillo and Vincent Lam show that some operational features of spacetime in General Relativity can provide salient information on time, independently of whether time is conceived of as emergent or fundamental. To do so, the authors make use of an approach that is known in the literature as “asymptotic reasoning,” that is, an approach that sacrifices the details of a given theory (or regime) to emphasize possible underlying patterns. This approach is tightly connected to notions such as universality and structural stability, where a system that is invariant under small perturbations is said to display a form of structural stability, and universality is invoked when distinct physical systems display similar dynamical patterns. After reviewing further details on the asymptotic reasoning approach, the contribution discusses the “operational” side of general relativity, in the sense of the theory as a guide to the interpretation of new observations from precision cosmology. From a discussion on the initial value problem in GR, the authors will retain two points on the evolution of spacetime dynamics. First, spacetime dynamics incorporates a mechanism that can smooth out the evolution of (a sector of) the dynamical degrees of freedom (therefrom the asymptotic reasoning and universality). Second, one can construct a temporal function that allows for an initial value formulation of spacetime dynamics in which there is structural stability with respect to some of the initial data. Afterward, the authors will use binary black hole (BBH) mergers as a case study to show how some structures underlying the theory are brought about by specific asymptotic limits. For example, time in the initial

value problem described above can be used for a parametrized description of the transition between initial and final states of the system. Such a parametrized description is shown to unveil some underlying structures through the separation between fast and slow degrees of freedom. Thus, asymptotic freedom should not only be considered in terms of mathematical methods but also as a possible exploratory tool. In addition, the case study of the BBH mergers shows the connection between dissipation/smoothing mechanisms, asymptotic reasoning, and some foundational features of general relativity.

Finally, in “On the Problem of Time in Asymptotically Safe Quantum Gravity,” Luca Zambelli focuses on asymptotically safe quantum gravity that tries to quantize general relativity on the same footing as any other field theory. According to Zambelli, this formulation of quantum gravity leads to a definite view on many foundational problems, including the problem of time and the probabilistic description of the evolution of space states of the universe. In his contribution, Zambelli provides a review of the recent developments and proposals toward some of the ultimate research goals in quantum gravity: background invariance, unitarity, cosmology, and the origin of scales. This in turn offers to the reader an overview of the questions of time and timelessness in another approach to quantum gravity and offers the possibility to compare it with other approaches presented in this volume.

The volume concludes with Part III, which is devoted to the cosmological implications of QG approaches and to relativistic cosmology. In Chap. 10, titled “Time, Spacetime, and F-Theory,” Enrico Cinti and Marco Sanchioni defend an interpretation of F-Theory for which there is only one temporal dimension, unlike the more traditional view that interprets both dimensions with negative signature as temporal. Starting from this alternative interpretation, the contribution explores the philosophical consequences with

respect to both the ontology of the theory and the issue of spacetime emergence in quantum gravity. The authors offer a definition of what constitutes a time dimension, that is, (i) a dimension that exists in spacetime and (ii) a dimension with a negative signature. Then, the authors argue that a spacetime dimension is characterized by the propagation of the graviton, which can be considered as a benchmark for determining spacetime in F-theory. After reviewing the most salient features of F-Theory, the contribution focuses on the traditional interpretation that takes both dimensions with negative signature as temporal dimensions. Afterward, the authors point out how the graviton, which represents spacetime as the excitation of the gravitational field, propagates along one dimension only, and thus conclude that the second dimension with negative signature of F-Theory is to be considered as non-temporal. The second part of the contribution discusses the ontology of the new interpretation and the philosophical consequences of the debate on the emergence of spacetime in quantum gravity. Since F-Theory reduces to type IIB string theory, one can use branes of different dimensionalities to describe different degrees of freedom in F-Theory. Therefrom, one can distinguish between matter, gauge, and gravitational fields based on the dimensionality of the branes, and thus account for the ontology of F-Theory in terms of such branes and their interactions. Nonetheless, the ontological status of branes remains unclear. For example, it remains an open question whether we should consider them as the objects that encode the fundamental degrees of freedom of the theory, and thus whether they should be considered as fundamental or composite entities. The final pages of the contribution point out another interesting aspect of the interpretation defended by Cinti and Sanchioni. Unlike most examples in the literature, spatiotemporal and non-spatiotemporal features interact together to obtain a configuration of F-

theory. Such an interaction is a new perspective on the emergence of spacetime in quantum gravity, and that is because most approaches seem to attempt to derive spacetime starting from non-spatiotemporal entities, rather than in conjunction with them. The topic is left to further philosophical discussions.

Another approach to quantum gravity with relevant cosmological implications is discussed by Daniele Oriti in "Hydrodynamics on (mini)Superspace or a non-Linear Extension of Quantum Cosmology." Oriti suggests a more general correspondence between hydrodynamics and cosmology, and a picture of the universe as a quantum gravity condensate. In his paper, he shows that the hydrodynamics of quantum fluids can be mapped to relativistic cosmological dynamics, and both share the same conformal symmetries, which can be unraveled via geometric methods in superspace. This picture is in fact realized also in some quantum gravity formalisms, such as group field theory and lattice gravity, in which an emergent cosmological dynamics can be extracted from the quantum dynamics of fundamental quantum simplices in a condensate phase. From the conceptual standpoint, these approaches assume the relational understanding of space and time, which makes superspace the natural arena for gravitational dynamics, as opposed to the "spacetime" manifold. These results suggest a careful examination of the interrelation among quantum gravity, the theory of quantum fluids, and cosmology, as well as the consideration of the impact that analogue gravity simulations in the lab can have for epistemological considerations that go beyond the problem of empirical adequacy of theories of quantum gravity and call for a deeper analysis of analogical arguments used in fundamental physics and cosmology.

In his contribution, "Arrows of Time in Bouncing Cosmologies," Marco de Cesare introduces the topic of the

arrow of time in physics and he focuses on the different arrows of time that can be introduced in bouncing cosmologies. He underscores how, at the background level, the cosmological arrow of time always points in the direction of expansion, thus away from the bounce. Furthermore, he shows that in bouncing scenarios one can define a thermodynamical arrow of time, pointing in the direction of horizon entropy growth. However, if the bounce is symmetric, the background thermodynamical arrow is unable to introduce any distinction between the two branches. He examines this case for a newly derived exact solution of the effective Friedmann equations of LQC describing a bouncing scenario with pressureless dust and a positive cosmological constant. This example shows that, in general, the analysis of the evolution of the cosmological background is not sufficient to draw any reliable conclusions about the arrow of time, leaving open many questions. Furthermore, when the evolution of scalar perturbations is taken into account, he discusses how the time arrow of inhomogeneities growth originates from initial conditions that break time-reversal symmetry. The author finds that a specific choice of (vacuum) initial conditions is made in the distant past, which amounts to a Past Hypothesis. This plays an analogous role to Penrose's Weyl hypothesis in standard cosmology in selecting a special initial state for the universe. However, de Cesare emphasizes that in order to fully address the arrow of time problem one should work in a full theory of quantum gravity, for which we lack a complete formulation at present. To complicate matters further, in quantum gravity the time arrow problem is also closely connected with the problem of time, i.e., the absence of time at a fundamental level. Indeed, if spacetime emerges dynamically from a pre-geometric structure, the problem of defining an "initial state" for the universe may require a reframing that makes no reference to the emergent spacetime geometry. Within

this framework, it would be interesting to investigate whether the initial conditions of the universe could be explained from first principles, for instance by means of a dynamical mechanism, or if a Past Hypothesis (i.e., a law-like prescription similar to the Weyl curvature hypothesis) is still required.

Finally, “Noether Symmetries in Quantum Cosmology: A Selection Rule for Observable Universes,” by Salvatore Capozziello stresses the relevance of symmetries as selection rules in cosmology. He focuses on the special role played by Noether symmetries in Quantum Cosmology and suggests that the former is a fundamental tool in order to assess the reliability of models. Capozziello considers minisuperspace cosmological models and looks for Noether symmetries in some minisuperspaces of extended theories of gravity in order to find out exact solutions. He shows that symmetries, and then conserved quantities, imply exact cosmological solutions. Thus, symmetries give a selection rule through which one recovers correlations among cosmological parameters, i.e., a criterion to select observable universes. Indeed, Noether symmetries allow to identify conserved quantities that select oscillating behaviours of the wave function of the universe. This in turn respects the Hartle criterion, according to which observable universes are solutions of dynamics as soon as correlations among physical variables are identified. Therefore, there is more to symmetries than a mere aesthetic function when describing the universe: they are an epistemic criterion to define physical systems.

Very much on the same vein, we believe that a reflection on time and timelessness and a new elaboration of the concept of atemporality are not just a fanciful game nor an unattainable goal, but could help us in identifying suitable invariants when describing physical systems. In conclusion, we hope this volume has conveyed the idea that time, spacetime, and a new meaningful notion of timelessness

will be of fundamental importance in shaping our physical theories. Indeed, the many contributions to this volume have clearly shown that any serious reflection on time in fundamental physics and cosmology calls for a discussion on the notion of timelessness. Yet, how this notion is to be characterized is still an open debate. We hope and believe that further interactions between philosophers, historians, and physicists will continue this investigation and further develop those concepts that interpret our universe and define our place within it.

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Acronyms

ADM Arnowitt, Deser and Misner

AS Asymptotic Safety

BBH Binary Black Hole

BH Black Hole

CDT Causal Dynamical Triangulation

CST Causal Set Theory

GFT Group Field Theory

GR General Relativity

QFT Quantum Field Theory

QM Quantum Mechanics

SM Standard Model of Particle Physics

TGFT Tensorial Group Field Theory

ΛCDM Lambda Cold Dark Matter—Standard Cosmological Model

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Part I
Time and Timelessness: Historical Outlooks

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1. Atemporality and the Origins of the Sempiternal Cosmos: Debates on Timeless Simultaneity within Platonist Cosmogonies

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Abstract
This chapter endeavors to explore the notion of atemporality within selected works belonging to the Platonist tradition. Beyond providing an overview of various facets of atemporality and highlighting their mutual relations, this chapter aims to investigate their role in a range of accounts of the world's origins. By focusing on the cosmogonical views elaborated by Platonists who deny that the cosmos is generated in time, such as Plotinus, Porphyry, Calcidius, and Proclus, I will dwell on a specific kind of atemporality, namely 'timeless simultaneity,' and shed light on its theoretical advantages in explaining the

demiurgic creation of the cosmos within a sempiternalist framework. Paradoxical as it may seem, within this perspective, the assertion that the Demiurge creates the cosmos at once does not conflict but, in fact, is fully compatible with the assumption that the cosmos has no temporal beginning, causally depends on a higher cause, and is always in a process of coming to be. As a result, a multi-layered taxonomy of atemporality, and especially the notion of ‘timeless simultaneity,’ enables Platonists adopting a sempiternalist stance to argue consistently that the cosmos is both ungenerated and created all at once, and to effectively explain in what sense it is so.

1.1 Introduction: Atemporality and Cosmogony

The aim of this chapter is to explore the notion of atemporality within selected works belonging to the Platonist tradition by showing its various facets and by investigating its role in an array of cosmogonical accounts. Although ancient philosophers mainly contrast time with eternity, their notion of atemporality is not straightforwardly identical with eternity. In fact, their account of atemporality has a wider extension and involves several manifestations that can be ordered in a multi-layered taxonomy. The first chief kind of atemporality is eternity conceived of as a timeless present, and thus distinguished from perpetuity, everlastingness, or sempiternity by virtue of its durationless nature. Additionally, atemporality includes instantaneity, regarded as a timeless juncture or boundary which allows for the transition from a given state to its opposite, such as the passage from life to death. Germane to the previous kinds of atemporality, without nevertheless being exhaustively identical with neither of them, is ‘timeless simultaneity,’

which characterizes processes wherein all parts of a given whole undergo change all at once.

Beyond providing an overview of these meanings of atemporality and highlighting their mutual relations (Sect. 1.2), in this chapter I will narrow the focus on the last kind of atemporality, namely 'timeless simultaneity,' thereby investigating its impact and role in a range of accounts of the world's origins (Sect. 1.3). By focusing on the cosmogonical views elaborated by Plotinus, Porphyry, Calcidius and Proclus, I will dwell on this kind of atemporality and cast light on its theoretical advantages in explaining the Demiurge's creation within a sempiternalist framework. Paradoxical as it may seem, within this perspective, the assertion that the Demiurge creates the cosmos "at once" does not conflict but, in fact, is fully compatible with the assumption that the cosmos has no beginning, causally depends on a higher cause, and is always in a process of coming to be. As a result, a multi-layered taxonomy of atemporality, and especially the notion of 'timeless simultaneity,' enables Platonists adopting a sempiternalist stance to argue consistently that the cosmos is both ungenerated *and* created all at once, and to effectively explain in what sense it is so.

1.2 Atemporalities in Antiquity: An Overview

In his discussion of "Zeilosigkeit" in the *Historisches Wörterbuch der Philosophie*, Thomas Sören Hoffmann emphasizes the scarcity of ancient Greek and Latin terms denoting "atemporal" (*achronos* and *intemporalis*) and describes them as a "*vox docta*."[1] The absence of a direct equivalent for "atemporality" in Ancient Greek, along with the infrequent usage of the Latin term *intemporalitas*, further complicates the exploration of this notion. While

ancient sources do not employ a singular term to encompass all kinds of atemporality, they explore this notion in various contexts and by using diverse terminology. Given the absence of a standardized vocabulary and a comprehensive taxonomy defining the various types of atemporality, the boundaries between them often remain ambiguous. Acknowledging potential overlaps among these diverse manifestations of atemporality found in ancient sources, it becomes possible to cautiously outline a preliminary taxonomy, to highlight their salient features, and to illustrate the examples provided by ancient authors to elucidate them.

For ancient philosophers, a first chief meaning of atemporality is eternity (*aiôn*, *aiônios*), which is frequently set in contrast to time.[2] Nevertheless, this assertion requires further refinement, as eternity is a puzzling notion that can be understood in multiple ways, not all of which necessarily involve atemporality. Atemporal eternity, or eternity proper, is commonly articulated by ancient and late ancient sources as a timeless present, devoid of duration and succession. Far from being an infinite stretch of time, eternity—often likened to a point[3]—is conceived of as a unified, unextended being which has no past nor future, but simply always is, as suggested by the alleged etymology of *aiôn*.[4] This understanding of eternity is widely attested at least from Plotinus onwards. While there is abundant evidence that Plotinus conceived of eternity in this way,[5] it is matter of debate whether such a meaning was available in earlier sources and there is no consensus on when precisely it appeared. Some scholars credit Parmenides with the 'discovery' of eternity as timeless present,[6] while others argue instead that this notion appears for the first time in Plato's *Timaeus*.[7] Moreover, evidence of eternity as timeless present is provided by Middle Platonist sources, as Plutarch[8] and Numenius[9]

tellingly illustrate. All in all, while it is challenging to precisely determine who first 'discovered' eternity as a timeless present, it is reasonable to assume an awareness of such kind of atemporality even before Plotinus. Subsequently, the portrayal of eternity (*aiôn*) as a timeless present becomes customary,[10] and is often accompanied by efforts to differentiate it from perpetuity, also referred to as sempiternity or everlastingness (*aidiotês*)—a distinction which becomes standard in the Middle Ages,[11] but is clearly foreshadowed in varied fashions and terminologies in late ancient writings.[12] Another important notion to express atemporality is that of instantaneity. A *locus classicus* on this subject is Plato's *Parmenides* (156c–157b), where the instant (*exaiphnês*) is described as that paradoxical nature (*physis atopos*), lurking between motion and rest, which is in no time at all (156d6–e1) and which allows for the sudden transition from a given state to its opposite. Since it is not in time but allows for changes taking place in time, the instant may be described, following Brisson, as a double threshold, horizontal between the past and the future, and vertical between time and eternity, forming a golden chain that connects time to the unchanging point of eternity.[13] In the tradition, a classical example of a process that occurs instantaneously is the *moriendi momentum*, which is represented as the sudden passage from life to death, i.e., from being to its contrary state of non-being.[14] Another example of instantaneous transition is the embodiment of the souls as it is described in the myth of Er, in which the souls, likened to shooting stars, are suddenly (*exapinês*) carried to their birth.[15]

However, ancient and late ancient sources bear witness to an additional dimension of atemporality, which I propose to call 'timeless simultaneity.' The latter closely relates to the two meanings of atemporality explored thus far, and in

certain instances, it even appears to overlap with them, without nevertheless being entirely identical to either. This is expressed by a cluster of terms, such as “together,” “simultaneous(ly),” and “at once” (*hama*; *homou*; *athroôs*), which are often used to qualify the grasping of a thing (through perception or intellection) and characterize processes of generation or alteration in which all parts of a whole come to be or change simultaneously. Typical examples of this kind of change are the freezing of water, visual perception, intellectual apprehension, and illumination, whether portrayed as lightning, an eclipse, or as the emission of light from the sun or fire.[16] Although these processes differ from each other, they are all taken to exhibit a common feature: the change involved therein does not affect some parts first and other parts later, but all parts at the same time. In this sense, these processes do not require a temporal progression for completion and can thus be said to occur atemporally, regardless from the fact that in nature they occur in time.

Having explored various facets of the ancient notion of atemporality and underscored the significance, alongside timeless eternity and instantaneity, of timeless simultaneity, I will now turn to examine the implications of this last kind of atemporality in ancient debates about cosmogenesis. Given its involvement in the generation or alteration of wholes, timeless simultaneity is exploited as a metaphysical tool within certain Platonists’ accounts of the creation of the world. Most importantly, and perhaps unexpectedly, this kind of atemporality proves to be crucial not only for creationists elucidating a punctual beginning of the cosmos, but also for Platonists defending the world’s beginningless and ungenerated nature.

1.3 Timeless Simultaneity in Cosmogony

Among the Platonists, it is particularly in Porphyry that timeless simultaneity, i.e., the kind of atemporality described as “at once,” proves to be crucial for explaining the Demiurge’s generation of the cosmos. This insight is gleaned from various sources reporting Porphyry’s perspective, namely Proclus, Simplicius and Philoponus.[17] As I will discuss more extensively in due course, these three accounts highlight different facets of Porphyry’s cosmogonical views, but all converge in suggesting that the significance of this form of atemporality in cosmogony serves as a benchmark of Porphyry’s doctrine. Admittedly, however, the view that the cosmos was created all at once can be traced back to authors predating Porphyry. Therefore, prior to our examination of Porphyry, let us briefly review some earlier views in which timeless simultaneity may appear to surface in cosmogonical discourse.

1.3.1 Foreshadowings?

The emphasis on the simultaneous creation of the cosmos can already be found, for instance, in Philo of Alexandria. In *De opificio mundi*, Philo connects the Bible’s account of creation with the cosmogonical story expounded in Plato’s *Timaeus*. In opposition to those who hold that the cosmos is uncreated and eternal, Philo explicitly asserts that the cosmos did indeed have a beginning (*Opif*. 7–12). On the other hand, when commenting on the opening line of *Genesis* (1.1), which states, “In the beginning God created the heavens and the earth,” Philo emphasizes that the expression “in the beginning” should not be understood in temporal terms (*Opif*. 26–27). In this work, he repeatedly stresses that God created everything simultaneously

(*hama*).[18] Philo's insistence on the notion of simultaneity, coupled with his assertion of a timeless beginning of the cosmos, may suggest that God's creation occurred as a punctual, albeit non-temporal, event.[19] Although Philo places considerable emphasis on the notion of simultaneity (*hama*) in explaining the creation of the cosmos by God, his exploration of atemporality implies less philosophical elaboration compared to later Platonists like Porphyry and Proclus. By asserting that God created everything at once, Philo does not delve into the role of a specific form of atemporality in cosmogony. Rather, his main concern appears to be the reconciliation of the *Genesis* account of creation in six days with the assertion that God was not "in need of a length of time" for the cosmos' creation, "not only in giving commands but also in his thinking."[20]

An intriguing precursor to Porphyry's doctrine is discernible in the works of Plotinus, who, to some extent, anticipates the significance of simultaneity in connection with cosmogony. Notably, while Plotinus extensively explores the notion of simultaneity (*athroos*, *athroôs*) in several places, he does not usually address it within a cosmological or cosmogonical framework, but primarily in relation to visual perception and intellectual apprehension.[21] Both sight and intellection, Plotinus explains, occur simultaneously in that the eye and the Intellect, when they focus on an object of perception or contemplation, do not fixate progressively on each single part of which it is made, but grasp it as a whole and all at once. It is noteworthy that Plotinus explicitly applies the feature of "at once" to the generation of the cosmos only on a single occasion (*Enn*. II 9, 12.12–23). While presenting his own cosmogonical account, Plotinus refrains from asserting this view explicitly, yet it becomes apparent through his critical examination of a view held by the Gnostics, who maintained that God began creation by

generating fire.[22] In addressing the problematic implications of this view, Plotinus contends that the divine cause of the world had a conception not only of fire but also of the cosmos as a whole, and indeed, he had to conceive the whole first. Consequently, Plotinus notes, the divine cause did not create fire first and then the other elements, but rather fashioned the cosmos as a whole all at once (*athroôs Enn*. II 9, 12.16). In further support of the earlier assertion, Plotinus explains why cosmogenesis is not procedural, but occurs at once, namely because it is a kind of natural generation. Just as in natural, i.e., biological generation, where "there is an outline and sketch plan of the whole living thing impressing the form on the menstrual fluid," in the creation of the world, too, matter is impressed with a sketch of the whole cosmos, containing all its elements simultaneously.[23] This stands in contrast to the Gnostics' view of cosmogenesis as a sequential process, where fire is generated first, followed by the other elements. All in all, while Plotinus may be seen as laying the groundwork for Porphyry's doctrine, he does not strongly thematize the aspect of "at once" atemporality when explaining how the cosmos came to be. It is only with Porphyry that this view is fully developed within the cosmological context, as we will see more closely in the next section.

1.3.2 Porphyry's Voice

The significance of simultaneity in Porphyry's cosmogony and its equation with atemporality becomes evident, sometimes more, sometimes less explicitly, through at least three sources. The first relevant passage in this respect is found in Proclus' *Commentary on the Timaeus*. Comparing God's creative act to that of an artisan who has no need of tools, Porphyry characterizes cosmogenesis as occurring both simultaneously and timelessly:

> Fourth and next is the section of [Porphyry's] arguments in which he shows that divine Intellect practises a mode of creation [which is performed] just by being and establishes [this] by a number of arguments. Even artisans [he says] need tools for their activity [only] because they do not have mastery over all [their] material. They show this themselves by using these tools to get [their] material ready for use by drilling, planing or turning it, all of which [operations] do not add form, but [merely] eliminate the unreadiness of the [material which is] to receive the form. The actual conformation [of the work], on the other hand, supervenes upon the material atemporally (*achronôs*) from the art once all inhibiting factors have been removed. And if there were no inhibiting factor in the case of [artisans] either, they [too] would add the form to the matter all at once (*athroôs*) and have absolutely no need of tools. And likewise the imagination too produces many effects in the body simply by its own action.[24]

The broader framework of this text is a discussion of Porphyry's arguments against Atticus,[25] who advocates for a literal reading of the creation story in the *Timaeus* and argued in favor of a temporal beginning of the cosmos. In the quoted passage, Proclus documents one of Porphyry's key objections: divine Intellect engages in creation simply by being. His mode of creation differs from the one performed by artisans, who require tools in order to overcome all obstacles related to their material and to make it apt to receiving the form. Once all hindrances are removed, matter receives form atemporally (*achronôs*) and at once (*athroôs*). Porphyry illustrates God's role as analogous to that of an artisan who, devoid of the need for

tools, can impress form upon matter in a timeless manner.[26]

Another significant text providing insights on Porphyry's application of the "at once" feature to the genesis of the cosmos can be found in Philoponus' treatise entitled *On the Eternity of the World against Proclus*. The passage in question immediately follows the well-known discussion of the multiple meanings of 'generated' (*genêton*), as expounded by Taurus and by Proclus.[27] The focal point of such distinctions is to highlight that being generated in time is only one among the various meanings that the term *genêton* can encompass. This clarification allows sempiternalist Platonists for the assertion that the cosmos has been generated[28] without necessarily implying that it had a temporal beginning. After dealing with Taurus and Proclus, Philoponus scrutinizes three supplementary meanings that are attributed to Porphyry.[29] The second meaning of "generated" is articulated as follows:

> In addition, Porphyry says that things which derive their existence from a [process of] generation and coming to be, for example a house or a ship or a plant or an animal, are also said to be generated. For this reason we do not describe a flash of lightning or a snapping of the fingers or anything else that exists and ceases to exist in an instant (*exaiphnês*) as generated; as Aristotle also says, all such things come to be without [a process of] generation and switch to non-existence without [a process of] decay. It is clear that nobody would hold that the world is generated in the sense of having come to be through a [process of] generation, for God brought all things into existence (*eis ousiôsin*) at once (*hama*) with the thought (*noêmati*). This being so, we shall have no need of

> this sense [of ‘generated’] in our investigation of Plato’s meaning.[30]

The meaning of generated (*genêton*) highlighted in this passage concerns things that come to be through a *process* of generation. Their coming to be unfolds over an extended period, involving duration, and progressing through various stages. This is the kind of generation that characterizes the coming to be of houses or ships, plants or animals. By contrast, examples like a flash of lightning and the snapping of the fingers are used to illustrate phenomena that come into being and cease to exist instantaneously (*exaiphnês* 149.3). In Porphyry’s view, the generation of the cosmos falls into this latter type of generation. Notably, when he shifts from the physical processes described in the examples to the metaphysical account of cosmogenesis, he stresses that this specific kind of “instantaneity” amounts to simultaneity; indeed, as Porphyry clarifies, God “brought all things into existence at once with the thought” (*hama* […] *noêmati* 149.9–10).

To gain a deeper insight into how Porphyry applies this distinction to cosmogony, let us examine Philoponus’ examples more closely. The first set of examples is nicely illustrated by Simplicius, who mentions animals, plants and houses as substances whose coming to be has a starting-point from some part of the object, i.e., the navel or the hearth (animals), roots (plants), foundations (houses).[31] He opposes these instances of coming into being with processes like freezing and illumination, wherein change does not start from a specific part, but concerns the whole and occurs all at once, devoid of any sequential progression. As you may recall, these are classical examples used to illustrate phenomena occurring at once. Simplicius’ examples of freezing and illumination correspond, in Philoponus’ passage, to the flash of lightning

and the snapping of the fingers, the former of which also elsewhere serves as a standard illustration of an atemporal phenomenon.[32]

In the light of these examples, it becomes evident that, according to Philoponus, Porphyry denies that the cosmos is generated in the sense that it is not brought into existence gradually, piece by piece, but it comes to be all at once through God's act of intellection. At this juncture, an intriguing question arises: is this simultaneous act atemporal? In this passage, instantaneousness and simultaneity (cf. *exaiphnês* and *hama*) do not imply, at least explicitly, atemporality. However, although Philoponus does not expressly mention atemporality, this hypothesis should not be dismissed. First, the contrast between what is brought into being at once and what requires a certain interval of time for generation suggests that God's act of intellection is atemporal, as it necessitates no process and implies no duration. Another argument supporting the idea that the Demiurge's creative activity, as referred to by Porphyry, entails atemporality is the inclusion of the example of lightning, often mentioned in other sources as an instance of a timeless process.[33] Indeed, the examples of lightning and snapping fingers may seem to capture atemporality only to a limited extent. Certainly, in a sense, they occur in time. Nevertheless, if read as metaphors introduced for illustrative purposes, these examples elucidate the nature of processes that do not involve temporal duration. Thus, they prove effective in explaining phenomena of atemporal simultaneous generation, including the origin of the world through God's intellection.[34]

While the passage we have just examined supports Porphyry's stance on simultaneous cosmogenesis without explicitly establishing the identity between simultaneity and atemporality, the connection between the two becomes

unmistakably clear in Simplicius.[35] After having distinguished the substances which have a starting point in some part of the object (e.g. animals, plants, house) from changes which occur all at once (e.g. freezing, illumination), Simplicius argues that the fact that the latter happen at once (*athroôs*) does not imply that they are timeless (*achronos*), spelling out that this was the way in which Porphyry understood it.[36] Disagreeing with Porphyry, Simplicius asserts that "at once" does not entail timelessness, but only indicates that all the parts change simultaneously. Accordingly, he notes, freezing and illumination "do not happen timelessly, but they have their beginning in time, at this special part of time, but all the parts undergo the effect together."[37] In other words, these changes are not necessarily atemporal for Simplicius, but involve processes in which all parts change simultaneously. Although Simplicius does not extensively delve into a discussion of Porphyry's perspective in this passage, he provides clear evidence of Porphyry's equation of the simultaneous with the atemporal, thus confirming what we have already encountered in Proclus' testimony.

All in all, within Porphyry's sempiternalist perspective on cosmogenesis, timeless simultaneity is helpful to better elucidate how the cosmos, albeit devoid of a temporal beginning, is brought about by the Demiurge. In that it depends on a higher cause, the cosmos has a causal, non-temporal beginning; viewed from the standpoint of its generation by the Demiurge, in turn, the cosmos is generated all at once and can be seen as the product of an atemporal creation. Thus, it is eternal and ungenerated on the one hand, yet the outcome of a timeless creation on the other. Accordingly, the perspective of a simultaneous, timeless creation is made compatible with sempiternalism, and the reference to this specific aspect of atemporality

proves to be fruitful to better grasp the derivation of the world from a higher cause.

1.3.3 Echoes

Porphyry's perspective on atemporality within cosmogenesis, as it has been outlined so far, has notable echoes in the other authors. In what follows, I will consider the reappearance and reappraisal of Porphyry's view in three selected, telling cases, namely those of Calcidius, Proclus, and Philoponus. Although Porphyry is not explicitly mentioned by any of these authors, his influence is detectable, to varying degrees, in all of them. Interestingly, while the first two share Porphyry's sempiternalist approach, Philoponus integrates the notion of a timeless, simultaneous beginning within a distinctively creationist framework.

A subtle echo of Porphyry's "at once" feature may be perceived in Calcidius, who is the author of an exceptionally influential translation and commentary on Plato's *Timaeus*. Throughout his commentary, Calcidius advocates for a metaphorical interpretation of cosmogenesis, and endorses a sempiternalist stance on *Timaeus*' creation story. He explicitly states that the world's origin "is causative (*origo causativa*), not temporal (*non temporaria*)," pinpointing the generation of the world to its ontological dependence on a higher cause, namely God.[38] In this context, although devoid of a temporal starting point, the cosmos can be said to be generated due to its reliance on a higher cause. Yet, this portrayal becomes more intricate when considering certain remarks that seemingly conflict with Calcidius' customary approach, which introduce an element of tension and prompt questions about the commentator's overall exegetical consistency. In a few cases, for instance, both in the

translation and in the commentary, Calcidius uses a vocabulary strongly connotated by temporalism.[39]

Another statement contributing to this complexity is Calcidius' assertion that the world was completed "at one and the same moment" (*uno eodemque momento*).[40] Interpreting this remark within the framework of Calcidius' usual perspective, i.e., his commitment to sempiternalism, makes it unlikely that he is advocating for a temporal beginning of the world based on a literal reading of the *Timaeus*. Now, the "at one and the same moment" Calcidius mentions does not necessarily imply a temporal aspect; furthermore, it is distinguished in the previous lines from the eternity of the intelligible paradigm and chiefly characterizes the generation of the whole cosmos. Hence, "at one and the same moment" is most likely to imply timeless simultaneity. This resonates with Porphyry's perspective[41]—a hypothesis which gains support when considering the very probable influence that Porphyry exerted on Calcidius.[42] If Calcidius, in the wake of Porphyry, understands *uno eodemque momento* in terms of a timeless simultaneity, then his remark is not at odds, but fully consistent with his broader account of a non-temporal origin of the world.

Embedded within a sempiternalist framework, Porphyry's distinctive "at once" perspective finds notable resonance in Proclus. As widely acknowledged, for Proclus the generation of the world must be understood in non-temporal terms, as an ontological dependance and as a non-durational process of derivation from a higher cause.[43] Since the Demiurge "belongs to the beings that always exist," Proclus notes, asserting that he creates "at one point in time" is absurd. Instead, it must be concluded that "he is always creating;" to suggest otherwise would imply idleness on the part of the Demiurge, which is impossible.[44] With that being said, let us now examine how Proclus

incorporates the “at once” feature within this sempiternalist framework. Following in the footsteps of Porphyry,[45] he characterizes the Demiurge’s ‘creation’ as an instantaneous event, drawing a parallel with the Sun’s instantaneous emanation of light:

> God brings forth all things all at once (*athroôs*) and throughout eternity (*diaiôniôs*). For it is through his very being and through his eternal thinking of wholes (*kata tên aiônion tôn holôn noêsin*) that he engenders all the things that result from him—the totality of things both hypercosmic and encosmic: intellects, souls, natures, bodies and matter itself. If you ask me, demiurgic creation exhibits this ‘all at once’ (*athroon*) aspect more than the Sun’s illumination does. In the latter case, the entire light proceeds simultaneously (*hama*) from the Sun. But even though the Sun imitates the Father through visible creation, this is clearly inferior to the Father’s eternal (*diaiôniou*) and invisible production. Therefore, as we said, though all things have come about from the act of creation all together (*homou*) and eternally (*aiôniôs*), nonetheless the order of effects is still preserved; for all things proceed all together (*athroôs*) and each with its own order since there was present in that which produced it an eternal intelligence and an order prior to the things that have been ordered. Hence, even if all things result at once (*homou*) from one thing, nonetheless some of them are of primary worthwhile others have a lesser value.[46]

Proclus does not use the term ‘atemporal’ in this passage, but resorts to the image of the simultaneous emanation of light from the Sun, thereby effectively conveying the atemporal nature of the demiurgic creation process.[47] This

allows Proclus to claim that the Demiurge creates the cosmos “all at once” (*athroôs*) and “throughout eternity” (*diaiôniôs*) while dismissing the hypothesis of a temporal cosmogenesis, both in terms of a specific point in time and in processual, durational terms. The cosmos can thus be said, from a viewpoint, ungenerated, eternal, and always in a process of coming into being; at the same time, from another perspective, it can be conceived of as a whole that derives from an instantaneous act of intellection by the Demiurge, a process that clearly resides outside the realm of time.[48]

In later times, a somewhat paradoxical revival of Porphyry’s perspective is evident in the writings of Philoponus. Remarkably, despite Philoponus outright rejecting the sempiternalist paradigm, he incorporates Porphyry’s instantaneous act of demiurgic creation into a creationist framework.[49] In this sense, he draws upon the Porphyrian timeless simultaneity to challenge the prevailing Neoplatonic account of an eternal act of creation. For instance, in one of his refutations of Proclus’ arguments, he contends that the Demiurge, when producing the cosmos, “brings everything into existence just by willing it and has no need of time,”[50] and elucidates this timeless creation by using standard examples, such as the emanation of light, visual perception, and noetic intellection.[51] The Porphyrian shaping of Philoponus’ perspective is also evident in some quotations reported by Simplicius, where God is portrayed as creating the heavens and the world not gradually, but immediately and timelessly (*amesôs*; *achronôs*), solely through his will.[52] As a result, while in Porphyry, Proclus (and arguably in Plotinus and Calcidius as well), timeless simultaneity coexists with a commitment to sempiternalism, Philoponus rejects the latter by advocating for a timeless, punctual act of creation by God.[53] Strikingly, he describes this act using

terminology and examples akin to those used by the thinkers he opposes.

1.4 Conclusion

In sum, the notion of atemporality manifests in a spectrum of meanings throughout antiquity and Late Antiquity. Notably, timeless simultaneity, sharing similarities with both timeless eternity and instantaneity, prominently figures in the cosmogonical narratives of various ancient authors rooted in the Platonist tradition. The emphasis on this kind of atemporality within cosmogony surfaces, though not yet fully developed, in Philo of Alexandria and, among Platonists committed unequivocally to a sempiternalist framework, in Plotinus. However, a comprehensive articulation of timeless simultaneity in explaining the Demiurge's creation is found in Porphyry, whose influence is perceptible in Calcidius and, more distinctly, in Proclus. Later, a rather strong echo of Porphyry's perspective is discernible in Philoponus, who, dismissing Proclus' arguments for the eternity of the world, draws upon Porphyry to argue in favor of a timeless beginning for the cosmos.

Remarkably, while in Philoponus timeless simultaneity aligns with creationism, in Calcidius, Porphyry and Proclus, this kind of atemporality operates within a sempiternalist framework. In their accounts, the Demiurge's simultaneous act of creation not only is fully compatible with the assumption of an eternal creation, both being atemporal, but also offers significant theoretical advantages in explaining the actual occurrence of cosmogenesis. As a matter of fact, this kind of atemporality allows these Platonists to effectively explain how the sempiternal cosmos was generated by a higher cause within a derivational causal model. Indeed, remaining firmly anchored within sempiternalism, these thinkers leverage

timeless simultaneity to capture the nature of the Demiurge's creative activity as a simultaneous act of intellection of the world, not *qua* sensible, but *qua* whole. In this way, their cosmos can be said to be both generated all at once and always in a process of coming into being.

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Footnotes

1 Hoffmann (2004, 1274–1275).

2 See e.g. Pl., *Tim*. 37d3–7; 38c1–3 and Plot., *Enn*. III 7, 1.1–3; 11.45–56.

3 See e.g. Plot., *Enn*. III 7, 3.19–23; VI 5, 11.14–21.

4 A commonplace in several sources is that *aiôn* (eternity) etymologically derives from *aei* (always) + *on* (being); see Ar., *de cael*. I 9, 279a25–28; Plot., *Enn*. III 7, 4.42–43; Syrian., *In Tim*. fr. 17 (Klitenic Wear) = Procl., *In Tim*. III, 15.11–16 (Diehl); Procl., *In Tim*. III, 9.15–16 (Diehl); Simpl., *In Phys*. 1155.13–16.

5 See at least Plot., *Enn*. III 7, 2.31–35; 3.11–38; 6.15–36. On Plotinus' account of time and eternity, see Beierwaltes (1967b); Plass (1977, 1–4); O'Brien (1985, 67–72); Graeser (1987); Strange (1994); and—more recently—Wilberding (2016, 32–45); Nikulin (2019); Chiaradonna (2022).

6 DK 28B8; cf. Owen (1966).

7 See esp. *Tim*. 37e–38a, where Plato notes that the forms of time of *was* and *will be* should not be applied to the eternal being (*tên aidion ousian*), whose only appropriate characterization is *is*. On the much-debated issue of whether Plato envisioned eternity as a timeless present, see at least Owen (1966); Whittaker (1968); Tarán (1979); Sorabji (1983, 108–112); Mohr (1986); Ilievski (2015).

8 See Plut., *De E* 393A–B, which Opsomer (2009, 159) has considered as the first “more or less unambiguous statement of the concept of eternity as durationless being” in Greek philosophy. *Contra* O’Brien (1985, 68).

9 Num., fr. 5 (Des Places). See, however, Burnyeat (2005, 155–160), according to whom the eternity referred to in this fragment does not involve timelessness but is rather to be understood as *present* being.

10 See e.g. Porph., *Sent*. 44.3; Calc., *In Tim*. chs. 25 and 106; Iambl., fr. 64 (Dillon); Procl., *El. theol*. prop. 55.

11 See Steel (2001).

12 Procl., *El. theol*. prop. 55, 52.30–54.3 distinguishes eternal from temporal perpetuity (*aidiotês*), and describes the former as “a perpetual steadfastness,” “having its existence concentrated in a simultaneous whole” (*homou pan*) and being “entire in itself,” whereas the latter as a process unfolding “in temporal extension” and composed of parts in succession (trans. by E.R. Dodds); on this passage see Phillips (1997, 177); Steel (2001, 6–7 and 11); Lang (2005, 164–165). A similar distinction is drawn in Procl., *In Tim*. I, 239.2–6 (Diehl): the temporal forever (*to aei to chronikon*), located “in extension,” stretches over the “the entire continuity of time;” the eternal forever (<*to aei*> *to aiônion*), by contrast, is “wholly all together” (*athroôs pan*) and located in the “now” (*nun*) (trans. by D.T. Runia, & M. Share). Cf. also *In Tim*. I, 278.9–11 (Diehl). On Proclus’ conception of time, eternity, and perpetuity, see ONeill (1962), Plass (1977, 4–15), Kutash (2009), and Vargas (2021). In similar vein, Simplicius distinguishes eternity proper from everlastingness in the world of becoming (*In Phys*. 1155.13–20). As Simplicius notes, the world is everlasting, “but not *qua* eternal,” in that its existence unfolds “throughout all time,” while that which is eternal (*aiônion*) “possesses all its substance, its capacity and its activity at once” (trans. by I. Bodnár et al). See also Olymp., *In Meteor*. 146.15–23. For further elaboration on these distinctions, see Steel (2001); Siniossoglou (2005); Wilberding (2016). These distinctions are arguably prefigured by Plotinus (*Enn*. III 7, 2.28–29; 5.15–18; 6.21–36); for more details refer to Graeser (1987).

13 Brisson (1970, 394). For further discussion about instantaneity, see e.g. Beierwaltes (1967a).

14 See Taurus *apud* Gell., *Noct. Att*. VII, 13.1–12 = T 13 (Petrucci). See also Iambl., *De an*. 36 (Finamore/Dillon) and Simpl., *In Phys*. 983.25–984.2.

15 See Pl., *Resp*. X, 621b2–4, along with Procl., *In Remp*. II, 352.23–353.7, who explicitly connects this passage of the *Republic* with Plato's account of the instant in the *Parmenides*.

16 See Ar., *Eth Nic*. X 4, 1174a14–b14; *De sens.* VI, 447a1–11; *Phys*. VIII 3, 253b23–26; Alex. Aphr., *Mant*. 143.4–35; 144.34–145.7; Plot., *Enn*. I 6, 3.9–16; II 8,1.39–43; IV 4, 1.19–20; V 5, 10.5–10; V 8, 6.7–12; *passim*; Porph., *Ad Gaur*. 11.3 (Kalbfleisch); Calc., *In Tim*. ch. 60; Procl., *In Parm*. 1237.24–28; 1237.36–41; *In Tim*. II, 102.7–16 (Diehl); Simpl., *In Phys*. 106.24–107.11; Ps.-Simpl., *In de an*. 131.16–132.7; Philop., *In Aet*. IV 4, 65.13–24; *In de an*. II 5, 297.2–10; Philop. *apud* Simpl., *In Phys*. 1173.3–8 (= fr. 129 Wilberg).

17 Procl. *In Tim*. I, 395.10–22 (Diehl); Simpl., *In Phys*. 106.24–107.11; Philop., *In Aet*. VI 8, 148.25–149.11, on which see below.

18 *Opif*. 13; 28; 67.

19 See Runia (1986, 426–433); (Runia 2001, 157). A similar view has been defended by Baltes (1976–1978, vol. 1, 32–38), who argues that Philo envisioned creation as a punctual, timeless event. Conversely, according to Sterling (1992, 40–41), Philo's account entails a *creatio aeterna* whereby creation did not actually take place, but signifies an ontological dependance of the cosmos on God. For further elaboration on Philo's views of eternity, time, and cosmogony, see Sorabji (1983, 203–209); (Runia 1986); (Runia 2001); Sterling (1992, 33–41); Calabi (2008); De Luca (2022, 109–110; 123).

20 *Opif*. 13 (trans. by D.T. Runia), on which see Runia (2001, 124–127).

21 See, already, Aristotle and Alexander of Aphrodisias, among others. Selected passages are provided above, fn. 16.

22 For an analysis of Plotinus' attitude towards Gnostic cosmology, see Chiaradonna (2015).

23 Plot., *Enn*. II 9, 12.17–23 (trans. by A.H. Armstrong).

24 Procl., *In Tim*. I, 395.10–22 (Diehl) (trans. by D.T. Runia, & M. Share, slightly modified).

25 Procl., *In Tim*. I, 391.4–396.26 (Diehl) = Porph., fr. 51 (Sodano).

26 On this passage, see Theiler (1933, 14); Baltes (1976–1978, vol. 1, 227–229); Chase (2011, 146; 2017, 343–344); Michalewski (2018, 127).

27 See, respectively, Philop., *In Aet*. VI 8, 145.13–147.25 and 147.25–148.7. For a comprehensive overview of this excursus see at least Phillips (1997). More specifically on Taurus' classification, see Petrucci (2018, 32–45; 230–233). On Proclus' meanings of 'generated,' see also Procl., *In Tim*. I, 279.30–281.13 (Diehl) along with Baltes (1976–1978, vol. 2, 19–23).

28 See Pl., *Tim*. 28b7 (*gegonen*) along with Baltes (1999 [1996]).

29 Philop., *In Aet*. VI 8, 148.7–149.16; a brief overview of these meanings is offered by Sodano (1962, 103 and 109–110).

30 Philop., *In Aet*. VI 8, 148.25–149.11 (trans. by M. Share, slightly modified) = Porph., fr. 36 (Sodano).

31 Simpl., *In Phys*. 106.24–27. Roughly the same set of examples features, in similar contexts, also in earlier sources. Both Atticus (fr. 4, Des Places) and Calcidius (*In Tim*. ch. 228) mention ships and houses, along with statues, in their respective discussion of the creation of the cosmos and of the World Soul. Further references to these examples in Alexander of Aphrodisias and Galen are provided by Baltes (1976–1978, vol. 1, 65 and fn. 181; 219).

32 See Porph., *Ad Gaur*. 11.3, Iambl., *De an*. 36 (Finamore/Dillon), and Procl., *In Parm*. 1237.36–41.

33 Cf. previous note.

34 As noted persuasively by Chase (2011, 115), Porphyry "seems to draw an analogy between these processes of instantaneous generation or change and God's creation of the universe. As in the case of these examples, the world did not have to undergo a process of generation in order to come being, but God brought it into substantification (*ousiôsis*) simultaneously with his thought (*hama noêmati*)."

35 See Simpl., *In Phys*. 106.24–107.11 = Porph., fr. 131 (Smith).

36 Simpl., *In Phys*. 106.33–107.2.

37 Simpl., *In Phys*. 107.2–5 (trans. by P. Huby, & C.C.W. Taylor).

38 Calc., *In Tim*. ch. 23, 74.17–19 (Waszink) (trans. by J. Magee). Calcidius draws a distinction between the birth of the works of nature, which occurs in time, from the works by God, which is causal and atemporal.

39 A reference to the temporal generation of the cosmos can be identified in Calcidius' translation of Pl., *Tim*. 28b6–8. Another related tension surfaces

within the commentary, between chs. 23 and 25. For further elaboration on this matter, see Galonnier (2009), Hoenig (2014), and Reydams-Schils (2020, 50–51).

40 Calc., *In Tim*. ch. 105, 154.18 (Waszink) (trans. by J. Magee).

41 According to a different interpretation put forward by Gersh (1986, vol. 2, 472), the expression *uno eodemque momento* indicates "that the creative process took place in the first instant of time which was itself not temporally extended." An alternative reading has been defended by Reydams-Schils (2010, 503; 2020, 56 and fn. 15), who equates *uno eodemque momento* with the "divine eternal present."

42 The extent of Porphyry's influence on Calcidius is matter of debate. For an exploration of Porphyrian echoes in Calcidius' treatment of time, eternity, and the origins of the world, see Waszink (1964, 37–82); Baltes (1976–1978, vol. 1, 177–184). Skeptical about Porphyry's impact on Calcidius is Dillon (1996 [1977], 403–404), who emphasizes instead Calcidius' deep-rooted affiliation with Middle Platonism. Also Reydams-Schils (2020, 172–190) expresses reservations about considering Porphyry a primary source for Calcidius and urges caution regarding an overestimation of such an impact. This position is partially revised in Reydams-Schils (forthcoming, section II), where the author is more inclined to acknowledge Calcidius' dependence on Porphyry in the light of the edition of the treatise *On Principles and Matter* (PM), which Arzhanov (2021) attributes to Porphyry. It cannot be ruled out that in this connection Philo of Alexandria—mentioned by Calcidius in *In Tim*. ch. 278—may have played a role too.

43 See Procl., *In Tim*. I, 276.30–282.22; 285.7–286.19; 381.26–383.22 (Diehl).

44 Procl., *In Tim*. I, 288.12–27 (Diehl) (trans. by D.T. Runia, & M. Share). More generally, on the 'Why not sooner?' argument and on the 'Idleness argument' see Sorabji (1983, 232–238; 249–252).

45 The Porphyrian inspiration of Procl., *In Tim*. II, 102.7 ff. has been acknowledged by Theiler (1933, 15).

46 Procl., *In Tim*. II, 102.7–22 (Diehl) (trans. by D. Baltzly, slightly modified).

47 That for Proclus "simultaneous" (*athroos*) and "atemporal" (*achronos*) may go hand in hand can also be gleaned from other passages, see e.g. Procl., *In Tim*. III, 21.32–22.13 (Diehl).

48 Cf. also Procl., *In Tim*. I, 281.14–282.22; III, 1.10–12; III, 322.1–11 (Diehl).

49 See Chase (2011, 147), who speaks of a "wonderful case of historical irony."

50 Philop., *In Aet*. IV 4, 64.22–26 (trans. by M. Share); cf. also *In Aet*. I 3, 6.2–12; I 6, 18.11–12.

51 Philop., *In Aet*. IV 4, 65.13–24. See also Philop., *In de an*. II 5, 297.2–10; Philop. *apud* Simpl., *In Phys*. 1173.3–8 (= Philop., fr. 129 Wilberg). Cf. also above, fn. 16.

52 Philop. *apud* Simpl., *In Phys*. 1141.24–30; 1173.11–13 (= Philop., fr. 115 and fr. 129 Wilberg). Porphyry's impact on Philoponus is also discernible in Philop., *In Aet*. IV 4, 66.25–27, where it is said that God creates solely through his will and without the need for any instrument. Compare with Porph. *apud* Procl. *In Tim*. I, 395.10–22 (Diehl), discussed above.

53 Incidentally, this might explain why Simplicius, a resolute opponent of Philoponus, diverges from Porphyry's equation of the simultaneous with the timeless (cf. Simpl., *In Phys*. 106.33–107.5, on which see above). For a discussion of the polemical debate between Philoponus and Simplicius on the eternity of the cosmos, see Wieland (1960) and Chase (2011).

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2. Time and Atemporality of Time in Hegel's *Naturphilosophie*

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Abstract
What is the meaning of atemporality in Hegel's philosophy? What is the relationship between philosophy and physics, according to Hegel's *Naturphilosophie*? And why should Hegel's reading of Plato's *Timaeus* be interesting for understanding both his idea of atemporality of time and the philosophical approach to the problem of the origins of the world? This Chapter addresses these questions by analysing some passages of Hegel's writings, from the *Dissertation on the orbits of the planets* (1801) up to the *Encyclopedia* (1817/1827/1830). Hegel tackles the concept of atemporality (*Zeitlosigkeit*) when he refers to the idea or to the ideal dimension as considered *in se* and *per se*, the question about the eternity or the beginning of the world emerges as an integral part of the problem. The Chapter

includes Hegel's reconsideration of Plato's *Timaeus* up to the arguments about the origins of the world and the notion of life in the second part of the *Enzyklopädie der philosophischen Wissenschaften im Grundrisse* devoted to *Naturphilosophie*.

2.1 Scientific Knowledge and Philosophy Around 1800

In his analysis of the "cosmological argument" in the *Critique of the pure reason*, Immanuel Kant questioned the link between the reflection about time and the discourse about the origins of the world: rational cosmology cannot avoid the antinomies of pure reason, because it deals with a notion of world as a totality which is not a phenomenal object (CpR, B 633–640). In Kant's first antinomy of the pure reason, the notion of time plays a crucial role: if the world has a beginning in time, the object is smaller than our concept and we must limit our conceptualisation; on the contrary, if there is no temporal beginning and the object is greater than our concept, we have an infinite regression in time and thus it is impossible to define the world as a whole.[1] Kant set the limits of pure reason when applied to objects lying beyond the empirical perception and generated a fracture between philosophy and the sciences of the universe, significantly determining the philosophical debates around 1800 in Germany. His dismissal of the connection between the notion of time and that of the world's origins had an impact on the thought of German *Naturphilosophen*, who aimed at reconsidering the relationship time-and-the-world from other perspectives.

In a manuscript comment on Plato's *Timaeus* (early 1794), the young philosopher Friedrich Schelling analysed the Platonic description of the beginning of the universe in *Timaeus* keeping an eye on the new theory of knowledge

provided by Kant's critical philosophy.[2] In his comment as well as in his writings of *Naturphilosophie*, Schelling surveys the notions of time, the world, and life, by showing the complex intertwining of philosophical investigation and scientific debates. His writings of the years 1797–1799 have been considered an example of the attempt of philosophers to reconcile speculative philosophy with the empirical scientific knowledge of the time (Follesa 2023). Schelling wanted to provide a theoretical support system in which every discovery and scientific theory would have been perfectly conciliated to one another. A fundamental assumption of Schelling's *Naturphilosophie* was the thesis of an original coincidence between nature and the universal spirit.[3]

On the footsteps of Schelling's, Georg Friedrich Hegel took part to the philosophical and scientific debates with his doctoral dissertation about *De orbitis planetarum*, discussed in Jena in 1801.[4] Many scholars noticed that, in comparison to the young Schelling, who manifestly had an interest for the recent discoveries in physics, astronomy, chemistry, and life sciences, Hegel's approach to the sciences has been essentially negative and characterized by a general detachment and misunderstanding of scientific theories. Moreover, scholars insisted on Hegel's claim about the superiority of philosophy above the sciences, his dismissal of empirical research and his remarkable errors concerning calculations and definition of physical laws.[5] His dissertation on *De orbitis planetarum* was considered as a proof of lack of scientific expertise and the literature underlined how his proposal opposed Newtonian physics (see Adler 1987). His rough 'mistakes' about the calculations that describe the position of the planets in the solar system have been criticized together with his return to Platonic or Aristotelian ideas.[6] Valerio Verra distinguished Hegel's "errors" from the misleading belief

that he had no interest for empirical and natural sciences (Verra 2002, 10–11), while Cinzia Ferrini investigated the historical context and demonstrated Hegel's acquaintance with scientific and methodological problems discussed by scientists of his time (Ferrini 1997, 150).

Indeed, Hegel's endeavour is to provide a conciliatory perspective for approaching philosophy and physics within a unitary discourse. He carefully examines the different methods and metaphysical perspectives that underlie each discipline and points out the philosophical meaning of the concepts that are "usual to the physics" (Verra 2002, 12). In the first part of his dissertation, Hegel's directs his argument against the "experimental philosophy," a term that he uses to refer to the English empirical tradition. The empirical (or experimental) philosophy aims at constructing a philosophical conception starting from perception and according to a mechanical perspective. As a result of this approach, philosophy is a mere abstraction and consists in fragmented, detached, unnecessary and finite parts of knowledge that have nothing to do with "true" philosophy, as an absolute and universal knowledge:

> True philosophy repudiates the principle of experimental philosophy, this principle being sought in a mechanics imitating nature on the level of dead matter and effecting a synthesis of absolutely different forces in any body. Now what pertains to the imitation of nature must be entirely cast aside in the cognition of nature itself, and in physics room must be given to neither chance nor whim (Hegel 1987, 286).

Any reader, who is not unfamiliar with philosophical thought, may easily connect these ideas to a cornerstone of the history of philosophy, *Timaeus*, where Plato distinguishes the "real" knowledge of immutable

intellectual objects from the opinion, which depends on the senses:

> We must begin by making the following distinction: What is that which always is and has no becoming, and what is that which becomes but never is? The former is grasped by understanding, which involves a reasoned account. It is unchanging. The latter is grasped by opinion, which involves unreasoning sense perception. It comes to be and passes away, but never really is (Plato, *Tim.* 28 d-e).

The reference to Plato[7] is helpful to understand Hegel's approach to the relationship between physics and philosophy in *De orbitis* and his definitions of time and atemporality as strictly related, on the one side, to the problem of the origins of the world and, on the other, to the concept of life, which underlies both Hegel's *Naturphilosophie* and Plato's conception of the world in *Timeaus*.[8]

2.2 Time and the World: Hegel's Reading of Plato

Hegel held several lectures on the history of philosophy in Jena in 1805/06, in Heidelberg 1816/1818 and in Berlin from 1819 onwards. In the *Vorlesungen über die Geschichte der Philosophie*, Hegel explored the steps of the history of philosophy, by paying special attention to the role of concepts and images in Plato's dialogues. The "sensible modes and sensuous images" adopted by Plato in his writings are "directed to representation and not to the concept in itself"; for this reason, mixing myth and philosophy expresses an "impotence of thought that does not yet know how to express itself in its own terms" and

aims at disseminating philosophical knowledge among the common people (Hegel 2011, 183). Despite the positive effect of reaching many people through the strength of images, the large use of myths in Plato's dialogues brings with it a dangerous implication: "many propositions are presented as doctrines of Plato that are really nothing of the sort," and this is particularly true with *Timaeus*. In this cosmological dialogue, Plato's description of the formation of the world is not expressed with purely philosophical words but through the so-called "representational mode" (see Plato *Tim*. 40d-41d; Hegel 2011, 183).

Contradictions may derive from the nature of mythological narrative which is based on images and analogies and disappear through a deeper examination of concepts in non-analogical terms (Hegel 2011, 212). The German philosopher attributes to Plato the idea that "with our eyes we distinguish everything – beginning with day and night, we have arrived at distinctions involving the months, the seasons, the years. This gave us knowledge of time and of history, and so we progressed to philosophy" (Hegel 2011, 184).[9] Despite his critique of the use of the representational mode, Hegel appreciates the Platonic philosophy as "the consciousness of the supersensible domain, of what is true and right in and for itself" (Hegel 2011, 185). Many meanings pertain to the term "philosophy"; in a Platonic sense, philosophy is the knowledge of the absolute, of what is *in se* and *per se*: what is universal and eternal is philosophical and thus it does not belong to temporality and change. Platonic philosophy belongs to atemporality and may be also considered an "idealism" which assigns to the doctrine of reminiscence the role of "foundation or general basis" of knowledge (Hegel 2011, 193). In Platonic philosophy, as it is expressed in *Timaeus*, the concepts of eternity and divine are not apart from the notion of "life":

> Whatever is undifferentiated is lifeless; hence the active, living, concrete universal is what differentiates itself inwardly but remains free in doing so. This freedom of differentiation consists in the fact that in its own other, in the many, in what is differentiated, the One is identical with itself. This constitutes the truth – the sole truth and point of interest for knowing – in Platonic philosophy; unless we know this about Platonic philosophy, we do not know the main thing (Hegel 2011, 202).[10]

According to Hegel's interpretation, there are two worlds—or two ways of approaching the problem of the world: "a divine world" which is juxtaposed to its copy, that is "the world where generation occurs, the visible world" (Hegel 2011, 214). The latter is generated together with time, the "self-moving image of the eternal" that proceeds "according to number." However, since time is the "immediate image of the eternal," it may be also considered as "authentic time," that is, as an eternal concept and an "eternal present," that "does not have future and past" (Hegel 2011, 214). On the other hand, the "real" parts of the world ("the sun, moon, and planets") delimit the "numerical relationships of time" so that ideal time can become a real being (Hegel 2011, 214). The concepts of time and the world have thus a two-fold nature: they can be understood as ideal concepts, which are known by the intellect, and as real beings, which are known by the sensible perception and, for this reason, are subject to change (Hegel 2011, 214). Hegel goes further and explains the notion of "eternal world" as a concept that entails two meanings: on the one hand, it implies the indeterminate idea of a world which is identical with itself; on the other hand, the world is "the other" if it is related to the idea of an immutable being.

Accordingly, the idea of the “world” includes the universal principle of the passage from an indeterminate world to another, thereby including the possibility of all changes. It is here that Hegel refers to the concept of “chora,” the “formless principle” and “wet-nurse (*trophos*) that sustains all things” and which is “receptive to any and every form” (Hegel 2011, 215). The idea of “world” encompasses the idea of “matter” or “passive matter” as clearly distinguished from the notion of “form” as an active principle (Hegel 2011, 215). Matter, as we read in the *Lectures*, is “universally substantial” and receives all the forms and geometric figures that Plato describes in *Timaeus* by appealing to a Pythagorean language (Hegel 2011, 214). Similar ideas are expressed by Hegel in a manuscript dated 1804/1805 and entitled *Logic, metaphysics, philosophy of nature* which focuses on the notion of “ether” or “absolute matter,” which corresponds to nature considered in itself and as a living being (Verra 2002, 16).

2.3 Atemporality and the Eternal Living Being

In the second part of the *Encyclopaedia of the Philosophical Sciences*, where Hegel exposes his *Philosophy of Nature*, the focus on time and the origin of the world reveals Hegel’s affinity with Plato’s cosmological narrative of *Timaeus*.[11] In an additional note to paragraph 247 of the section “Begriff der Natur” (Concept of Nature), Hegel’s includes an important distinction between the idea of the world conceived as a whole and the “natural world,” or, more simply, “nature.” The latter is the physical world that is subject to change and juxtaposed to the universal concept of the world as an eternal totality. This distinction essentially corresponds to two different ways of

understanding the world—as already expressed by Plato in *Timaeus*—empirical perception and pure intellectual intuition. According to Hegel, clear distinction of this kind is preliminary to solve the issue of the *eternity of the world*:

> The question of the eternity of the world (this is confused with Nature, since it is a collection of both spiritual and natural objects) has, in the first place, the meaning of the conception of time, of an eternity as it is called, of an infinitely long time, so that the world had no beginning in time; secondly, the question implies that Nature is conceived as uncreated, eternal, as existing independently of God. As regards this second meaning, it is completely set aside and eliminated by the distinctive character of Nature to be the Idea in its otherness. As regards the first meaning, after removing the sense of the absoluteness of the world, we are left only with eternity in connection with the conception of time (Hegel 2004, 15).

If it is possible to speak of the "eternity of the world," it means that the world has, or had, no beginning in time and has always existed. This statement implies, however, its full independence from God and, therefore, the identity between God and nature. Nature, in its being, is however "the other" of God, that is, God "in its otherness," or, to use Hegel's words, God "between the idea and nature." This definition of the world coincides with the first meaning which is the world in its absoluteness, the world as a universal concept. According to this position, the world had no beginning in time, and this implies a more detailed analysis of the concept of time and of the relationship between time and eternity:

> Eternity is not before or after time, not before the creation of the world, nor when it perishes; rather is eternity the absolute present, the Now, without before and after. The world is created, is now being created, and has eternally been created; this presents itself in the form of the preservation of the world. Creating is the activity of the absolute Idea; the Idea of Nature, like the Idea as such, is eternal (Hegel 2004, 15).

The juxtaposition between the concept of eternity (what has neither a "before" nor an "after" and belongs to an "absolute present") and that of the world, as a finite being which has been necessarily created (because this is implied in the idea of being finite), appears neat at a first glance. Hegel however introduces the distinction between the world as a physical reality and the world as a universal concept and, in this way, he can establish a correspondence between the notion of "eternity" and that of "creation"—a correspondence that sounds very closely to the Neoplatonic definition of "eternal creation," or "eternal emanation" of the finite world from the One. The world as a universal concept encloses the idea of a continual creation or derivation from the absolute Idea, which is characterized by an eternal activity. Exactly this idea of eternal creation goes hand in hand with Hegel's conception of an "eternal life," conceived as an "eternal creative process" and attributed to God. According to Hegel's interpretation, this definition agrees with Plato's definition of God in *Phaedrus* and *Timaeus*. The question about the eternity (or, alternatively, about the beginning) of the world has thus a double answer, or we can react to it, with Hegel, from two different perspectives. A sophisticated reasoning would demonstrate the impossibility to provide "a plain, positive answer" to the question "whether the world has, or has not, a beginning in time." This is because, Hegel continues, "a

plain answer is supposed to state that either the one or the other is true" and, he concludes "the plain answer is, rather, that the question itself, this 'either-or,' is badly posed" (Hegel 2004, 16). Thus, let us consider now two perspectives that enable us to approach the problem of the "eternity" of the world and the problem of its "beginning":

> In the question whether the world or Nature, in its finitude, has a beginning in time or not, one thinks of the world or Nature as such, i.e., as the universal; and the true universal is the Idea, which we have already said is eternal. The finite, however, is temporal, it has a before and an after; and when the finite is our object we are in time. It has a beginning but not an absolute one; its time begins with it, and time belongs only to the sphere of finitude (Hegel 2004, 15–16).

The problem of time is not completely separated from the problem of the beginning of the world (in time). According to this view, the world, as such, implies the idea of a beginning, a creation, which distinguishes it from the concept of God; and, considered as a universal concept, the world has been eternally created. At the same time, the world can be considered as a finite being: one may ask the question if, in this case, the (finite) world has or does not have a beginning *in time*? The world, as such, is "created." However, this does not immediately mean that the world has a beginning in time. As a totality, indeed, the world has a timeless dimension, is an atemporal (eternally) created being and lives in its eternal present standpoint, the world is an eternal being, that is, a universal idea, which is eternally present and eternally living. Understanding this kind of world, the universal concept of the world, pertains to philosophy, as a "timeless comprehension, of time too and of all things generally in their eternal mode" (Hegel

2004, 16). On the other hand, there is the natural world, everything that we perceive through the senses as a part of nature in its variety and dynamism: this world is situated on a finite, spatial and temporal dimension and thus belongs to time. The connection between being and knowing, between ontology and gnoseology, which is the heart of Plato's narration in *Timaeus* is another crucial element for understanding the problem of time and of timelessness in Hegel. It is precisely the change of perspective that determines, according to Hegel, the possibility to affirm something about the beginning of the world in time:

> If we are talking of the finite, then we have both a beginning and a non-beginning; these opposed determinations in their unresolved and unreconciled conflict with each other, belong to the finite: and so the finite, because it is this contradiction, perishes. The finite is preceded by an Other, and in tracing out the context of the finite, its antecedents must be sought, e.g., in the history of the earth or of man. There is no end (*Ende*) to such an inquiry, even though we reach an end (*Ende*) of each finite thing (*Endliches*) (Hegel 2004, 16).

The fact that we have two possible answers to the problem of the beginning of the world depends on the fact that "we are talking of the finite," which has "both a beginning and a non-beginning." For its intrinsic contradiction the finite begins and perishes, being preceded and followed by other finites. And if we go back in the chain of the finites, looking for the first beginning of the series, there is no such thing as a beginning or end, since we are dealing with finite things:

> Time has its power over the manifoldness of the finite. The finite has a beginning, but this beginning is not the First; the finite has an independent existence, but its immediacy is also limited (Hegel 2004, 16).

It does not make any sense to investigate the very beginning of the world as a finite being, since we would risk lingering curbed in an endless regressive process. The finite world has a beginning, but also a "before" and an "after" of this beginning, because we are dealing with "relative" notions:

> When ordinary thinking forsakes this determinate finite, which is preceded and followed by other finites, and goes on to the empty thought of time as such, or the world as such, it flounders about in empty ideas, i.e., merely abstract thoughts (Hegel 2004, 16).

It is impossible to determine the concept of the world as such or of time as such starting from the consideration of the empirical world that is finite and has a beginning in time. By following this path, namely by establishing an arbitrary beginning of the (infinite) series of finite, we just have "merely abstract thoughts" that are "empty." Hegel's notion of time, the notion of what is before and what is after, agrees with Plato's statement in *Timaeus* 28 b, that is the "creation" of time as a copy of eternity according to number carries out an ordering and regulating function in the physical world. On the other hand, according to Hegel, we can talk about another kind of time, the infinite time, i.e., time as universality and totality, which has nothing to do with the world of sensible things, and does not derive from the consideration of the succession of things in the

natural world but is related to that of representations and concepts:

> Infinite time is only an idea,[12] a going into the beyond, which remains infected with the negative; a necessary idea so long as one is confined to a consideration of the finite as finite. However, if I pass on to the universal, to the non-finite, I leave behind the standpoint where singularity and its alternate variations have their place. In our ordinary way of thinking, the world is only an aggregate of finite existences, but when it is grasped as a universal, as a totality, the question of a beginning at once disappears. Where to make the beginning is therefore undetermined; a beginning is to be made, but it is only a relative one (Hegel 2004, 16).

Changing the perspective and considering the infinite not as whole deriving from a series of finite items, but as an immediately given whole, leads to a change in the way in which Hegel considers the notion of "life" that is closely connected to that of "eternity." Eternal life is not the "natural life"; an immortal living being does not belong to time but has a life beyond time, in a timeless, logical, ideal dimension—a crucial concept for German Idealists. An immortal living being does not belong in the physical, sensible world, but in the ideal one as a living idea. Thus, in Hegel's view, there is no juxtaposition between philosophy, as the science of absolute concepts, and the scientific investigation of the finite world.

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Footnotes

1 On Kant's antinomy and regressive series, see De Bianchi (2013, 2015), Failla (2013), Proops (2014).

2 The reader can refer to Schelling's manuscript on *Timaeus* (1794) published in Follesa (2023), Schelling (2008) and Schelling (2016) for different editions and commentaries.

3 For a discussion of the main tenets of Schelling's philosophy of nature, see Heckmann et al. (1985), Moiso (1990), Follesa (2014), Zammito (2018).

4 See Hegel (1986, 1987).

5 On this subject, see Harris (1974), Ferrini (1985, 1991, 1996), Horstmann and Petry (1986), Ihmig (1987), Petry (1993), Moiso (1997).

6 See for instance Closs (1908), Oeser (1970), Shea (1986), Ruggiu (2013) and Ziche (2013).

7 For the importance of Plato for German Idealism, especially as concerns the world-soul and the problem of the world's origins, see for instance Viellard-Baron (1979), Greipl and Schönberger (2017), Beierwaltes (2011), Helmig and Marongiu (2020).

8 See Hegel (2011, 207). The problem of time has been widely discussed among Hegel's scholars and philosophers in the last century. From Heidegger's *Sein und Zeit* up to the recent studies, Hegel's philosophy of time has been considered, in particular, from the perspective of Hegel's *Phenomenology of Spirit* and his philosophy of history, where the notion of time plays a crucial role (Brauer 1982; Bodei 1984; Rametta 1989; Luckner 1994; Chiereghin 2000; Apple 2008; Bouton 2000 and McCumber 2014).

9 Hegel slightly changes this paraphrase by putting "history" in place of "the nature of the universe."

10 In the *Parmenides* Hegel could find the definition of "pure thoughts" as something "alive" and "not dead." Thus, "pure thoughts" are "self-moving and active," and their activity "consists in making themselves into their own other" (Hegel 2011, 206).

11 See Hegel (2004, Sect. 247, additional note, 13).

12 In German we find the term 'Vorstellung' that is translated in English with 'idea;' here the Italian translation as 'representation' helps the reader in clarifying the meaning.

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3. Not Even Ideal. Kant on Absolute Time and Gödel's Rotating Universes

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Abstract
In this Chapter, the concept of time is analyzed in a particular set of solutions of Einstein Field Equations that was proposed by Kurt Gödel (1906–1978) in 1949 and 1952. The rotating universe model has been largely inspired by Gödel's reading of Kant's *Metaphysical Foundations of Natural Science* (1786) in which Kant provided an argument for the relativity of motion of reference frames and for the rejection of absolute time in cosmology. After presenting Gödel's analysis of Kant's notion of time in his manuscripts, I shall show which aspects of Kant's philosophy had an impact on Gödel's view. Finally, I shall conclude by showing that Gödel's

critical reading of Kant's theory knowledge did not take into account his late work and therefore could not comment on core ideas of Kant's late cosmology embodied by the *dictum* 'forma dat esse rei.' Indeed, in the *Opus postumum* Kant appealed to the concept of cosmic aether as hypostatized space and provided an argument for distinguishing the notion of cosmic local time, which is necessary for the description of the observable universe, from an a priori account of the cosmos as a whole.

3.1 Introduction: Gödel's Appraisal of Kant's *Phenomenology*

In a volume devoted to time and timelessness in fundamental physics and cosmology, it seems natural to mention Kurt Gödel's contribution to debates in both physics and philosophy. Together with Luitzen E. J. Brouwer (1881–1966), Henri Poincaré (1854–1912) and Hermann Weyl (1885–1955), Kurt Gödel (1906–1978) might well be considered one of the greatest polymaths of the 20th century, who heavily contributed to the foundations of logic, mathematics and physics. He had robust philosophical views that exerted great influence on his scientific and mathematical work. In particular, the notion of time always attracted Gödel's attention because it is a subject that connects philosophy with the sciences and he dealt with it in several occasions during his life.[1] Gödel's model of a rotating universe and his reflections upon Kant's philosophy highlight interesting questions surrounding time and timelessness in relativistic physics and cosmology.[2] Thus, this Chapter investigates Gödel's interpretation (even if it would be more correct to define it as an application) of Kant's philosophy of space and time.[3] In *The modern development of the foundations of*

mathematics in the light of philosophy (1961), Gödel concludes:

> I believe that precisely because in the last analysis the Kantian philosophy rests on the idea of phenomenology, albeit in a not entirely clear way, and has just thereby introduced into our thought something completely new, and indeed characteristic of every genuine philosophy—it is precisely on that, I believe, that the enormous influence which Kant has exercised over the entire subsequent development of philosophy rests. Indeed, there is hardly any later direction that is not somehow related to Kant's ideas. On the other hand, however, just because of the lack of clarity and the literal incorrectness of many of Kant's formulations, quite divergent directions have developed out of Kant's thought—none of which, however, really did justice to the core of Kant's thought. This requirement seems to me to be met for the first time by phenomenology, which, entirely as intended by Kant, avoids both the death-defying leaps of idealism into a new metaphysics as well as the positivistic rejection of all metaphysics. But now, if the misunderstood Kant has already led to so much that is interesting in philosophy, and also indirectly in science, how much more can we expect it from Kant understood correctly? (Gödel 1981 [1961], 385; 387)

As we shall see, whereas Gödel's appraisal of Kant's 1786 *Metaphysical Foundations of Natural Science* (MAN)[4] is clear, the same cannot be said for the analysis he provided of Kant's theoretical philosophy in general. This depends on Gödel's dissatisfaction with the idea that a progress in the knowledge of the essence of the world is limited to phenomena and marked by the impossibility of knowing the thing in itself as the *correlatum* affecting our senses.[5]

Nevertheless, in the passage quoted above from Gödel (1981 [1961]), the reference to Kant's phenomenology is not to be understood in Husserlian terms or at least not directly and simply as it stands. Gödel is rather alluding to Kant's *Phenomenology*,[6] which is represented as the most genuine legacy of Kant's mature philosophy of nature discussed in the MAN.

In what follows, I shall first resume Kant's arguments for relative motion, absolute space and relative time as spelled out in 1786. In Sect. 3.3, I shall briefly present Gödel's solution to Einstein's equation and comment on the influence of Kant's philosophy of time. In Sect. 3.4, I shall present the 'new' cosmological picture advanced by Kant in the *Opus postumum* and present notions of time and timelessness that he used in natural science. Before concluding, in Sect. 3.5, I shall confront Gödel's interpretation of Kant's philosophy and provide evidence for the fact that Kant shaped his late philosophy to make the universe as a whole an indirect object of experience thereby asserting that the thing in itself is—at least with respect to the form—determinable from a further standpoint than that described in the 1780s.

3.2 Kant's Critique of Absolute Time (1786)

Let us now take a step back and resume Kant's results obtained in the MAN with regard to space, time and motion. Surprisingly, the *Phenomenology* is one of the most underestimated parts of Kant's system (De Bianchi, 2022b). Gödel had the merit of understanding well before Michael Friedman's analysis that this section of the work deserved careful attention since it presented an argument for the relativity of motion of inertial frames (Friedman, 2013) that, as Gödel underlined, represents an embryonic form of

Einstein's relativity principle (1905) and, more importantly, offers a clear account of a relational time when conceptions of absolute time were dominating pre-relativistic physics.[7]

In recent work, I have argued that Kant's notion of relational time has been employed to make sense of Euler's equations of rigid bodies (De Bianchi, 2022b).[8] Indeed, in 1786 Kant aimed at grounding a metaphysics of nature devoid of fallacies to account for any extension and development of Newtonian mechanics by means of the assumption of the law of antagonism in all community of matter through motion based on the principle of reciprocal interaction (*Wechselwirkung*). The latter has the relevant function of reducing and unifying all appearances under an idea of Reason (De Bianchi, 2022b). This law embodies the great advancement in Kant's ontology and idea of simultaneity that is reported in the second edition of the *Critique of pure Reason* (CpR). As Kant states in the *Introduction* to the second edition of the CpR: "the simultaneity of substances in space cannot be cognized in experience otherwise than under the presupposition of an interaction among them; this is therefore also the condition of possibility of the things themselves [*Dinge selbst*] as objects of experience" (CpR B258). Kant's argument for this conclusion is that just as empty or absolute time cannot be perceived and thus used to determine succession or simultaneity, neither can absolute space be "an object for our possible experience" (CpR A214/B261) and thus either the simultaneity of two objects, or the specific place of a single object, can only be determined through the category of community and the dynamic community (*commercium*) or reciprocal influence of all substances.[9]

Let us now consider more in detail the implications that this law has in the MAN. In the *General Remark to Phenomenology* Kant comments on the general use of three concepts in natural science: motion in relative (movable)

space; motion in absolute (immovable) space; relative motion in general as excluding absolute motion. The basis (*Grund*) of these three concepts is the idea of absolute space. The relationship between these concepts of motion and the idea of absolute space should be understood as a ground-consequence (*Grund-Folge*) relation. Thus, absolute space, as an idea of reason can never be a possible object of experience, but it is necessary to ground the three kinds of motion:

> Absolute space is therefore necessary, not as a concept of an actual object, but rather as an idea, which is to serve as a rule for considering all motion therein as merely as relative; and all motion and rest must be reduced (*reducirt*) to absolute space, if the appearance thereof is to be transformed into a determinate concept of experience (which unite all appearances). (MAN AA 04:560)

This argument relies on the ideality of space set forth in the first Critique, since for Kant external absolute space can only be generated *ad infinitum*.[10] From this follows the general rule that all physical motion (and rest) can only be relative and never absolute, that is, matter can be thought in motion (or at rest) only in relation to another portion of matter and never with respect to mere empty space. Therefore, not only absolute motion is impossible without any relation of one matter to another but even the notion of absolute rest loses any meaning (see MAN, AA 04:559–60). Thus, absolute space becomes and idea of the reference frame through which not only all relative motions, but also reciprocal empirical interactions are determined as relative. In MAN, Kant claims that “absolute space is *in itself* nothing and no object at all,” but amounts to the condition for thinking of an indefinite process of considering ever more extended relative spaces (MAN, AA

04:481–82). Moreover, Kant portrays absolute space as the rule for considering all motion and rest therein merely as relative (MAN AA 04:560). This conception is closer to Euler's (1752, 186) rather than Newton's. Indeed, differently from the latter, Kant applied the concept of central force to the universe as a whole and in particular to its internal structure by building up a hierarchy of systems starting from our solar system and further including galaxies and clusters of galaxies.

As a second point of departure from Newton, Kant clearly attributes only a transcendental status to absolute space, which cannot neither be taken as a substance nor being physically intuitable. Finally, Kant claims that gravitation is only one particular instance of the concept of central force and that the universal law that Kant has in mind is the one from which Newton's third law is derived, i.e., the law of antagonism in all community of matter through motion (see MAN, AA 04:563). In order to obtain experience of matter through the empirical concept of motion, one needs a double determination of time (as form of intuition and as formal intuition). Indeed, to determine acceleration in agreement with Euler's equations of motion of rigid bodies, other two elements which give us a necessary universal determination of bodily motion are required: the translation of the center of mass and the rotation about the center of mass. In other words, to prove that the Earth is revolving upon its axis or completing its revolution around the Sun we do not have to observe the starry heavens, but rather the measurement apparatus must be put within the systems of which we want to determine the relative motion. To prove the Earth's revolution about the Sun, one has to consider the Sun-Earth system as two body problem:

> There is thus no absolute motion, even when a body in empty space is thought as moved with respect to

> another; their motion here is not considered relative to the space surrounding them, but only to the space between them, which considered as absolute space, alone determines their external relations to one another and is in turn only relative (MAN, AA 04:562).

It is worth mentioning that the Newton-Euler equation fits nicely with the observations made by Kant in the MAN. In classical mechanics, the Newton–Euler equations describe the combined translational and rotational dynamics of a rigid body. This formalism relates the motion of the center of gravity of a rigid body with the sum of forces and torques (or moments) acting on the rigid body. In other words, it embodies in a very sophisticated way the scope of objectively determining material bodies through actual relative motions, namely actual relative motions extend Newton's laws to all appearances, such as co-moving bodies, be they elastic, rigid or fluid. Therefore, Kant's *Phenomenology* aimed at encompassing the scope of Euler's Mechanics.

Differently from Euler, however, Kant dismissed the idea of absolute time and removes it from the picture, even if he clearly thinks that the objective representation of time as a formal intuition is fundamental to give meaning to the measurement of acceleration and therefore to connect the plurality of co-moving bodies. Indeed, Kant supports the empirical reality of time, i.e., its objective validity in view of all objects that may ever be given to our senses, but according to transcendental idealism it is not an absolute reality, and if one abstracts from the subjective conditions of sensible intuition, it is nothing at all (see CpR B52–54). Time is not even ideal without intuition. This aspect deserves careful attention. In the second edition of the *Critique of pure Reason*, Kant draws an important

distinction between space as formal intuition and as form of intuition:

> Space, represented as object (as is really required in geometry), contains more than the mere form of intuition, namely the comprehension of the manifold given in accordance with the form of sensibility in an intuitive representation, so that the form of intuition merely gives the manifold, but the formal intuition gives unity of the representation. In the Aesthetic I ascribed this unity merely to sensibility, only in order to note that it precedes all concepts, though to be sure, it presupposes a synthesis. (CpR, B160–161 footnote)

Contrary to the geometrical representation of space, time must undergo an indirect objective representation through motion (see CpR, B154–56). In other words, only by means of measurement the subjective representation of time acquires an objective meaning, it is so to speak made 'tangible' in physics, e.g., by measuring acceleration. The *Phenomenology* shows us that time can be represented as object through co-moving reference frames rather than by means of an abstract motion of a space. However, if co-moving reference frames are at stake, it means that their velocity can be measured relatively to one another and that their relative magnitude can be established with respect to the observer. In other words, time can be represented as object, but not, for instance, as a single cosmic time. Time is always relative and represented as effective only relatively to the observer measuring other objects, i.e., reference frames, astrophysical objects and so forth. This move conforms Kant's theory of matter to Euler's equations but creates an asymmetry between space and time. This is due to the fact that an idea of absolute time would lead to a concept of absolute acceleration and the impossibility of

comparing reference frames. In connection to the results of the *Phenomenology* regarding the three types of motion this would also lead to consequences contrary both to the universal law of antagonism in all community of matter through motion and to Kant's own ontology. We have now the necessary tools to compare Kant's and Gödel's views in the next Section.

3.3 Gödel's Rotating Universe(s) and Kant's Concept of Time

Gödel's solution of Einstein's field equations describing a rotating universe has been subject of a vast literature and studies that engaged, among others, Sir Roger Penrose, George Ellis and Stephen Hawking (see Hawking and Ellis, 1973). The solution was published in July 1949 under the title "An Example of a New Type of Cosmological Solutions of Einstein's Field Equations of Gravitation" and described a non-static stationary universe that is rotating upon its axis endowed of a specific metric dubbed 'Gödel metric.' The model provided a solution that did not include cosmological time that on the ground of Weyl's postulate was rather introduced by Robertson (1933):

> All cosmological solutions with non-vanishing density of matter known at present have the common property that, in a certain sense, they contain an 'absolute' time coordinate, owing to the fact that there exists a one-parametric system of three-spaces everywhere orthogonal on the world lines of matter. It is easily seen that the non-existence of such a system of three-spaces is equivalent with a rotation of matter relative to the compass of inertia. In this paper I am proposing a solution (with a cosmological

term $\neq 0$) which exhibits such a rotation. (Gödel, 1949).

Gödel's solution served as toy model to show that Einstein's field equations included solutions of a non-static universe, thereby enlarging the analysis of relativistic cosmology. Gödel also provided a broader analysis of time in relativistic cosmology. In particular, he enumerates the properties of the solution, which are properties of the four-dimensional space defined by the solution itself:

- The space is homogeneous: there exists a one-parametric group of transformations of this space into itself which carries each world line of matter into itself (any two world lines of matter are equidistant), and it has rotational symmetry.
- The totality of time-like and null vectors can be divided in positive and negative vectors so that a positive direction of time can be introduced in the solution and "a temporal orientation can be defined for the world line of every (real or possible) particle of matter or light" (Gödel, 1949). This means that there is *a rule for determining* for any two neighboring points on the world line which one is *earlier than the other*. Nevertheless, there no exist any uniform temporal ordering of all point events agreeing in direction with all these individual orderings.
- This straightforwardly leads to the property of Gödel's model according to which it is impossible to assign a time coordinate t to each space-time point in such a way that t always increases when one moves in a positive time-like direction (this characteristic has the famous implication, which I shall not discuss in this contribution, of closed time-like lines admitted by Gödel's model and implying the theoretical possibility to travel into the past or influence the past in specific world lines of matter).[11]

- The model has the property of excluding the existence of three-spaces that are everywhere space-like and intersect each world line of matter in one point.

However, the most relevant characterization of Gödel's model for the present Chapter is the following. Gödel defines $\sum$ as "any system of mutually excluding three-spaces each of which intersects every world line of matter in one point" (Gödel 1949). Then he concludes that it must exist a transformation which carries the four-dimensional space and the positive direction of time into itself, but that does not carry $\sum$ onto itself. The upshot of all this is that absolute time does not exist, namely there no exist a time definable without reference to individual objects, i.e., a galactic system etc. Finally, the ninth property states that matter everywhere rotates relative to the compass of inertia with a definite angular velocity defined by the mean density of matter ρ and Newton's gravitational constant κ, reading $2(\pi\kappa\rho)^{\frac{1}{2}}$.

The latter is a property that further stimulated the analysis of the function of the cosmological constant in analogy with inertia and is still used for pedagogical purposes in current physics. However, Gödel's model is not considered a faithful representation of the expanding universe, rather it is taken as a powerful cosmological toy-model and a solution of the field equations. Einstein himself agreed with its elegance, but immediately reacted to it by underlying the necessity of empirically testing it. Without entering in the physics of the solution, one can easily see that it has several conceptual and philosophical implications that are worth being considered in the light of both his reflection and Kant's. In particular, it is worth underlying that Gödel not only dismissed the notion of an absolute cosmic time, but he also endorsed an account of relativistic time that echoes the definition of time that Kant used in the MAN.[12] This emerges in the work titled

“Rotating Universes in General Relativity Theory” (1952) and in “Some observations about the relationship between theory of relativity and Kantian philosophy”, in which he writes:

> […] The impossibility of defining any absolute time (among or besides the various relative ones) in the empty space-time scheme of special relativity theory (upon which the foregoing considerations have been based) does not exclude that matter and the curvature of space-time produced by it, *if the structure of the world as a whole is taken into account*, may enable us to determine some objectively distinguished ordering of all events to which the properties contained in our intuitive idea of time could consistently be attributed, and compared to which the various observed times would appear as something like systematic errors due to motion of the observers. This view is supported by the fact that in all known cosmological solutions (i.e., relativistically possible structures in the large of non-empty worlds) such an “absolute world time” really can be defined. But nevertheless the conclusions drawn above can be maintained because there exist other cosmological solutions for which a definition in terms of physical magnitudes of an absolute world time is demonstrably impossible. If, however, such a world time were to be introduced in these worlds as a new entity, independent of all observable magnitudes, it would violate the principle of sufficient reason, insofar as one would have to make an arbitrary choice between infinitely many physically completely indistinguishable possibilities, and introduce a perfectly unfounded asymmetry. Therefrom it follows that the possibility of a determination of an absolute world time, where it exists at all, is certainly not due

> to the laws of nature (which are satisfied in all cosmological solutions), but only to the special distribution and motion which matter has in those instances. A lapse of time, however, would have to be founded, one should think, in the laws of nature, i.e., it could hardly be maintained that whether or not an objective lapse of time exists depends on the special manner in which matter and its impulse are distributed in the world. (Gödel, 1946/9 B2, 237–238, emphasis is mine).

Gödel is here referring to his 1952 work on rotating universes in which he defines the conditions for his solutions to allow for the representation of cosmic time or "world time." The latter is by no means to be considered as the result of an absolute slicing of "nows," but it is a cosmic relativistic time that is admitted by Gödel's solutions for an isotropic but *not homogenous* rotating universe. More importantly, Gödel's argument relies on considerations involving symmetry of the laws of nature and, in this respect, he is using symmetry in a transcendental sense: the a priori condition for the possibility of cosmic time must derive from the symmetric laws of nature and must respect logical a priori principles, e.g., the principle of sufficient reason. This is what I dub 'Gödel's Kantian take' on cosmology. Indeed, his cosmological models 'exploit' the symmetries of the laws of nature as a priori principles determining the condition of possibilities and the modality (in a Kantian sense) that we can express in our judgments regarding entities, such as cosmic time. These symmetries of physical laws by themselves do not imply any cosmic absolute time nor an absolute independent time interval between events. Before analyzing in more detail Gödel's debt to Kant, let me first introduce his view on Kant's philosophy of time.

3.3.1 Gödel in Between Kant and Einstein

Gödel's interest in comparing Einstein's relativity and Kant's philosophy of space and time is mentioned in the letter he wrote to his mother on November 7th, 1947:

> I was asked to write a paper for a volume on the philosophical meaning of Einstein and his theory; of course, I could not very well refuse. I am also not sorry that I have accepted and chosen this theme [on the relationship of Kant to relativity theory], because *the problem* has always interested me and its fundamental investigation has in addition led to purely mathematical results. (Wang, 1987, 38)

In order to spell out the terms of *the problem*, let us consider the pages containing reflections upon Einstein's theory and Kant's philosophy. In "Some observations about the relationship between theory of relativity and Kantian philosophy" (1946/9–B2) Gödel immediately clarifies that he is not an adherent of Kantian philosophy in general,[13] but that he finds remarkable that there are several aspects of Kant's view of space and time (especially time) that are compatible with relativity and that it would be a mistake to conflate, as it happened in the past, Kant's notion of time with Newton's. I want now to highlight a second point that Gödel considers relevant, that is the Kantian conception of space and time in the treatment of rigid bodies. In *Some observations* he writes:

> Kant however denied the objective existence not only of space and time, but also of spatial and temporal relations (B42a), and therein exactly consists the novelty of Kant's view. That time and space have no existence independent of and beside the things was asserted already by Leibniz. (Gödel, 1946/9 B2, 238)

This point is extremely important for Gödel, since it vindicates in the most appropriate terms the subjective nature of time, time intervals and time relations in general. We have already seen in Sect. 3.2 that for Kant there is no absolute time in the universe: time can only be objectively represented when associated to relative motions, but a regulative idea of absolute space must be presupposed to generate a model of nature in agreement with Euler's equations of motion. The upshot of all this is an asymmetric treatment of space and time in cosmology. In other words, cosmic timelessness must be admitted, since the relative time of reference frames is never independent from the observer,[14] whereas the ideality of space means that absolute space cannot be ruled out at all from the picture. It is an ideal of absolute space, a regulative idea of reason that must be considered *as if* it were objectively existing independently from the subject and as a container of absolute spatial relations for any pair of material objects. Gödel notices this asymmetry in treating space and time even in relativistic physics. Indeed, he states:

> But for space the situation is different. The spatial relations directly observed and measured, it is true, also have no absolute meaning, since they are different for different observers. But, in contradistinction to time, there also exist for any two material objects (i.e., things persisting in time, not events) absolute spatial relations and they lead to a structure (see below) not so very different from that of intuitive spatial relations. (Gödel, 1946/9 B2, 238–239)

Contrary to temporal relations, spatial relations among two things or material objects can exist in terms of "true spatial distance" of two points of these two material objects. The rules of these absolute spatial relations are the schema of

all possible motions of rigid bodies and this schema is independent of both the observer and the reference frame:

> These movements can be described by the world lines of all points of these bodies, and these world lines in their turn are described by their coincidences with or relative positions to the material points of the actually existing bodies. This schema is what may be called the objective physical geometry of the world, of which the various geometries existing for various observers are only different aspects. (Gödel, 1946/9 B2, 239)

This passage not only represents Gödel's interpretation of Kant's *Phenomenology,* but also its application and translation in the framework of relativistic physics. There is however one point to clarify and that concerns Kant's doctrine of space and time as spelled out in the *Prolegomena* and the *Critique of pure Reason*. Indeed, Gödel notices, "these absolute spatial relations, it is true, are not strictly isomorphic with the corresponding ones of intuitive space (insofar as, e.g., they are non-Euclidean)" (Gödel, 1946/9 B2, 240), given that the concept ("finitness") of straight line in Riemannian geometry is completely in disagreement with our spatial intuition. Gödel assumed that for Kant spatial relations of pure intuition correspond to Euclidean space. This interpretation naturally led him to stress the difference between 'Kant and us,' namely for Gödel relativity marks a step beyond the ontology and the epistemology of transcendental philosophy because we are closer than Kant in grasping the real structure of the world. In other words, we make things in themselves more intelligible than previously thought and advance in the knowledge of the real geometric structure of the world. This move necessarily includes a "kinematic" of

scientific progress and is compatible with the idea of a multi-layered epistemology:

> In the present imperfect state of physics, however, it cannot be maintained with any reasonable degree of certainty that the space-time scheme of relativity theory really describes the objective structure of the material world. Perhaps it is to be considered as only one step beyond the appearances and towards the things (i.e., as one "level of objectivation," to be followed by others). Quantum physics in particular seems to indicate that physical reality is something still more different from the appearances than even the four-dimensional Einstein-Minkowski world. T. Kaluza's fifth dimension points in the same direction. (Gödel, 1946/9 B2, 240).

However, Gödel's interpretation is not entirely correct. Kant never identified the pure form of outer intuition with Euclidean space. He only stated that space has three-dimensions, whereas the pure form of inner intuition, i.e., time, has one dimension. The reason why Kant did not identify space as form of intuition with Euclidean space is trivial: for Kant, Euclidean geometry, like any conceivable geometry, represents space as object and is the result of what Kant identified as a synthetic construction of concepts. In other words, the form of outer intuition alone can never be identified with Euclidean space because the latter is the result of a complex construction of concepts in intuition that imply that the principles of the pure understanding are at work and generate through imagination suitable schemas to apprehend the manifold of inner sense according to rules, e.g., through algebra or geometry, depending on the synthesis involved (see Rechter, 2013). Euclidean geometry and its objects or axioms cannot be directly intuited, they can only be

constructed, according to a priori rules of synthesis. This in turn shows that Kant's view of geometry is far more sophisticated and to think that all that can be said with regard to space and time is contained in the *Prolegomena* and the *Transcendental Aesthetic* of the first Critique is a mistake that we can forgive to Gödel, but that we should avoid after 200 years of Kant scholarship.

3.3.2 On the Impact of Kant's Notion of Time on Gödel's Solution

We are now in a position to evaluate the impact of Kant on Gödel's work on the solution to Einstein field equations as presented in Gödel (1949).[15] Recall that for Kant geometric construction presents concepts *in* intuition and the latter has two pure forms: space and time. The forms of intuition can be represented objectively in geometry, in particular it is the case of space, and it can be done via construction of concepts according to schemas and in agreement with the principles of the pure understanding.

By rephrasing Leibniz's and Wolff's rules of ontological ordering, Kant embeds spatial and temporal relations within a new framework. In particular, in the *Critique of pure Reason* and late works, Kant describes them as the rule of positioning "beside and after one another" (*neben und nach einander–iuxta ac post se invicem ponendo*) the manifold in apprehension.[16] To order beside one another parts of space, e.g., adjacency, juxtaposition and so forth, is what characterizes a priori relations of the parts of the pure form of outer intuition: principles of action or principles of motion must always assume rules of ordering and express relationships of adjacency between two points and then describe how it is conserved or under which condition it is not the case. This is translated in Gödel's solution when he states that:

> After a direction of time has been introduced in this way, a temporal orientation is defined for the world line of every (real or possible) particle of matter or light, i.e., it is determined for any two neighboring points on it which one is earlier. On the other hand, however, no uniform temporal ordering of all point events, agreeing in direction with all these individual orderings, exists. (Gödel, 1949)

Thus, Gödel is here translating Kant's philosophy in the construction of his solution: temporal orientation for the world line of each particle can be introduced, but it cannot imply a uniform temporal ordering of all point events. What is left to us is just the determination of the ordering between two points and this ordering is spatial in the first place. Then, one can interpret this spatial relation in terms of temporality and thus construct a temporal direction in apprehension, but this is possible only because a representation of space is assumed. Gödel (2000 [1952]) offers an even subtler analysis of Kant's philosophy and silently vindicates it when it comes to the denial of closed time-like lines. In it, Gödel writes:

> For these solutions, also, the nonexistence of closed time-like lines is equivalent with the existence of a "world-time," where by a world-time we mean an assignment of a real number t to every space-time point, which has the property that t always increases if one moves along a time-like line in its positive direction. (Gödel, 2000 [1952], 1424)

The world-time is a number assigned to each (but not all) space-time point and has a direction. Recall that for Kant time is the form of ordering or the mode in which we relate the content of inner intuition to apperception. The rule of ordering temporal relations is to be *after* one another (*nach*

einander). This point is crucial. The ordering does not determine which element comes before and after the other, but only *after*. In other words, the form of pure intuition of inner sense has a direction, it points towards the positive direction that we call 'future.' The upshot of all this is that there exist solutions of Einstein field equations that not only can spell out the rules of spatial but also of temporal ordering that pertain to intuition. In Kantian terms this means that they are *givable* rather than *thinkable*. Therefore, not only cosmic time, but also time relations are not even ideal, where by 'ideal' I mean something that can become a concept of Reason, an idea, in Kantian terms.

Finally, another important aspect that characterizes Kant's legacy for Gödel's work concerns the definition of time intervals relative to astrophysical moving bodies. This is something that permeates not only Kant's transcendental philosophy but his entire cosmology as set forth from 1755 onward. Indeed, Kant generalized the application of Newton's law of gravitation to structures far beyond the solar system, including galaxies and nebular clusters. In Kant's view, the universe must be modelled as a system of systems, more precisely as a system of co-moving systems and sub-systems. This is also in line with his treatment of relative motion in the MAN and with the necessity of presupposing a regulative idea of absolute space when treating the universe as a whole as constituted by different densities of matter. This in turn led him, just like Gödel, to imagine the geometry of the universe: Kant used a spherical geometry to represent the classical expanding universe. Gödel, on the contrary, considering the different geometry and metric of spacetime admitted an elliptic shape as the one compatible with the homogeneous non-isotropic rotating universe. Nevertheless, not only both denied the existence of absolute time, but they also converged in defining relative time with respect to moving

bodies such as galaxies that were not homogeneously distributed.[17]

3.4 Time and Timelessness in Kant's Late Natural Science and Cosmology

In Sect. 3.2, I reminded that Kant draws a distinction between formal intuition and form of intuition. The distinction between a subjective and an objective representation of space and time is fundamental to understand how Kant meant to apply his transcendental philosophy to the scientific enterprise. The fact that one can objectively represent space, as it happens in geometry, naturally poses the question regarding the objective representation of time. More in general, the question arises when one wants to find the *pendant* (*Gegenstück*) of such objective representation in processes or entities other than motion. It must be noticed that Kant never attained a final answer to this question and in a sense the notion of time as formal intuition constituted a problem in his system more than a solution. We witness Kant's attempts at reconstructing a refined view until the last days of his life. In the late 1790s he effectively merged space and time and talked about them as *absolute unities*:

> All phenomena of matter and their moving forces are connected to the entire universe because space and time are absolute units. One can therefore assume a general principle of their reciprocal interaction, which exists in real relationships to one another, and experience is not possible other than insofar as every object is thought of with every other in this interaction and is assumed to be given a priori in appearance.[18] (OP 22:340, translation is mine)

In this passage, space and time are seen as a unique whole and are defined in terms of absolute units. A caveat is in order here: to talk about absolute *units* is not the same thing as talking about absolute space and absolute time. Again, Kant is here referring to his doctrine set forth in the CpR according to which spaces or times can always be intuited as embedded in *one* greater space and time.[19] Now, the interesting point is that in the manuscripts of the *Opus postumum*, not only spaces and times are thought together, but also the greater units space and time are unified and together characterize the aether as the realization of a unit of all possible interactions in the universe that can be given a priori in intuition. In the late period, Kant defined the aether as hypostatized space (OP 22:221) and he did not represent it as an absolute ideal space as he did in the MAN, but he talked about it in terms of a postulate that makes the reciprocal effective interaction among phenomena possible.

Kant attempted at giving several proofs and definition of the aether as the all-pervading and self-oscillating original cosmic matter filling the universe and making the latter empirically perceivable (OP 21:219-220). It is also defined as the original material (*Urstoff*) endowed with attraction and repulsion, namely he understood it as the receptacle of the dynamical forces of matter in contrast with the mechanical ones (centrifugal and centripetal forces). Nevertheless, in the late period (1796–1804), we witness once again Kant's denial of cosmic absolute time, but this did not prevent him from admitting a systematic treatment of the universe and its content: a world time can be admitted in Kant's system and it is when an account of the origin of the universe is needed. In order to provide a cosmogonic description, one must consider the beginning of the (relative) motion of matter in time, which in turn signifies the condition of possibility to locally define a

chronological order (OP 22: 195–198).[20] However, the cause of the first motion of the original matter is to be found in a virtual agitation that expresses a priority that cannot be understood in terms of temporal succession, but as a necessary transition that is thought in analogy with the concept of an end (*Zweck*): an internal ground for the generation and growth of organisms has driven the universe to evolve and the aether to expand, and since internal structures were generated and moved one with respect to the other, the idea of cosmic time toward the future can regulatively be admitted in cosmology (OP 22:272). The transition to the physical evolving universe, in turn, is not just virtual but also temporal. In the manuscript of the *Opus postumum*, indeed, there is a sense of temporality that Kant describes and that is worth mentioning: the instant.

Contrary to Plato, who in the *Parmenides* tried to underscore how atemporality admits a form of instant that is out of the temporal series (see De Bianchi, 2022a), Kant 'temporalizes' the notion of instant (*Augenblick*) and thinks of it as limit (*Grenze*). In particular, Kant thinks that this notion of temporality is necessary to describe phenomena like phase transitions in crystallization and the generation of matter. However, this amounts to the denial that time intervals are something depending on constitutive principles of/in natural processes and phase transitions (OP 21:270; 22:275), because Kant claims that the transitions or the switches do not happen in a time lapse of which we can measure a duration by comparing relative motions, rather we must mathematically represent the transition as a limit that we calculate through the method of fluxions. Thus, precisely because time is something subjective, instants can still acquire a meaning in so far as they can signify infinitesimal intervals to be measured and calculated by means of fluxions. However, we can never

have (and we do not need) a direct objective experience of an instant when talking about the generation of matter.

3.5 Kant's Hypostatized Space

Before concluding, I would like to draw attention on the fact that in the 1790s, Kant added several notes and reflections surrounding the scholastic dictum *forma dat esse rei* (the form makes the thing real) and the concept of causality. The dictum represents pretty well the metaphysical take that Kant endorsed to justify his late cosmology. Indeed, not only he believed that one can a priori determine and classify the moving forces of matter according to the formal logical rules of the understanding, but one can even a priori postulate the material form, i.e., the aether as hypostatized space that allows one to divide the universe in spatio-temporal regions possessing an internal dynamics. This move is not only compatible with making the idea of local world time possible together with the reciprocal interaction among bodies, but it also enables one to think of the universe as an evolving totality in analogy with natural ends, e.g., organisms.

What is givable (*dabile*) and thinkable (*cogitabile*) with respect to the form is completely determined a priori and this seems to be a natural outcome of Kant's universal law of reciprocal interactions and the necessity of representing systematic unity of reason in natural science. Nevertheless, there is a further point to be highlighted: since the whole of experience is made possible through the postulate of the aether, Kant found (or he believed so) the ground for defining a priori the thing in itself, i.e., the whole of experience, from a different perspective (*Gesichtpunkt*). This perspective was extremely relevant for him because it was the formal and material condition of possibility for unifying the objective and subjective representations of

space and time: the aether or hypostatized space is the material ground for any possible outer and inner intuition of our sensibility, the form according to which our senses can be affected by the external world and the formal and material condition of perception.

Thus, in the *Opus postumum*, we notice a further step towards the unification of Kant's forms of intuition of space and time together with the attempt at showing that time is subjective, but it can be embedded in a superior objective representation together with space: the aether. When saying that we could have had further interpretation in store for Kant's philosophy, nobody is more correct than Gödel in having understood the great intuition that Kant had in denying the objective and absolute nature of time. However, for Kant, time is not an illusion. For to say that something is subjective does not equate to say that it is not real, rather time is not existent as an absolute substance and it is not an independent physical object. Furthermore, the mathematical construction of times allows one to define time operationally as formal intuition in physics, and it does so in many different ways, as it is evident even in physics today.

3.6 Closing Remarks

This Chapter provided a link between the historical and the systematic part of this volume by showing the impact of Kant on Gödel's rotating universe model. In particular, it discussed the denial of absolute time in both Kant and Gödel, by including Kant's *Phenomenology* within the picture. This in turn led us to compare their views and to show that Kant admitted a further standpoint according to which time can also be considered real (*wirklich*) when associated to space, and as objective when considered as formal intuition. Moreover, in the 1790s, Kant embraced a

view according to which the notion of hypostatized space, the aether, unifies space and time into a whole. The metaphysical stance advanced by Kant is sum up by the scholastic dictum ‘forma dat esse rei,’ according to which the total complex of spatio-temporal phenomena must be ontologically grounded on the complete system of moving forces of matter to make the idea of the evolution of the universe possible. This in turn offers a different and richer reading of Kant’s philosophy of time and cosmology, and inevitably marks a difference with relativistic physics, but for reasons different from those identified by Gödel.

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Footnotes

1 For a detailed overview of Gödel's philosophical notes, see Crocco and Engelen (2016).

2 For a recent discussion of Gödel's universe and conception of time, see Kiefer (2023).

3 Kant is to be mentioned among the most important philosophers that had an impact on Gödel's scientific production. Gödel was engaged with the study of Plato, Aristotle, Augustine of Hippo, Thomas of Aquinas, René Descartes, Gottfried Wilhelm Leibniz and Georg Wilhelm Hegel, as well as with

contemporaries, such as Gottlob Frege, Giuseppe Peano, Bertrand Russell, Luitzen Brouwer, and Rudolf Carnap (see Crocco and Engelen, 2016).

4 References to Kant's texts follow the pagination of the Academy edition (AA). References to the *Critique of pure Reason* are abbreviated as CpR, followed by the A/B editions pagination, whereas those to the *Opus postumum* are abbreviated as OP. Translations are from the Cambridge Edition of the Works of Immanuel Kant unless stated otherwise.

5 For a detailed discussion of Gödel's idealism, see Parsons (2010).

6 The *Phenomenology* is the fourth and last Chapter of the MAN in which Kant wants to harmonize the mathematical and philosophical representation of time, space, and motion with both the foundations of cosmology and the end of natural science in general (see MAN, AA 04:477).

7 It would be misleading to consider Kant's foundations of natural science as anticipation of Einstein's theory. However, there are at least three aspects of Kant's work that are conceptually relevant to assess why Gödel pointed out analogies between the work of Kant and Einstein: the negation of absolute rest, the negation of absolute time, the relativity of motion of inertial frames.

8 The study of the latter together with the reflection upon the notion of force occupied Gödel's philosophical reflections that he explicitly connected to Kant's philosophy (see Gödel 1949b, 238–239).

9 As we shall see, this position will constitute the core argument for the development of Kant's late cosmology and natural science (see Sects. 3.4 and 3.5).

10 For Kant, both space and time are defined as *quanta continua*, and they are infinite given magnitude in intuition, e.g., their parts always presuppose a greater whole in which they are contained. Gödel strongly criticized the view of space and time as infinite *quanta continua*, but his critique is made from the logical standpoint and it does not fit in the Kantian scheme. Indeed, for Kant, general logic has nothing to do with space and time but only with pure

functions and the form of inferences of superior faculties, e.g., Understanding, Reason and so forth.

11 Gödel never thought that these curves were geodesics with necessary physical meaning. They are mathematical possibilities, see Audureau (2021).

12 Dorato (2002a) devotes careful attention to a comparative analysis of Kant's and Gödel's views of time, but does not consider the *Phenomenology*, whereas Dorato (2002b) argues that arguments supporting the ideality of time based on Gödel's own 'rotating' solution to Einstein's field equation fail.

13 Gödel's text states: "In order to prevent any misunderstanding, I wish to say right in the beginning that I am not an adherent of Kantian philosophy in general."

14 Gödel writes: "Now as to time this view of Kant's has been verified to a large extent by relativity theory insofar as, in general, for some of the pairs of events no absolute temporal relation whatsoever exists in relativity theory, and for the remaining ones [there exist relations of an entirely different nature and structure]" (Gödel, 1946/9 B2, 238).

15 It should be noted that Gödel's reading of Kant also relied on his study of Leibniz, which in turn led Gödel to focus on a relational reading of Kant's notions of space and time.

16 Note that for Kant "before" is something thinkable but not givable in terms of temporal relations in intuition. Furthermore, this means that the representation of space grounds the temporal apprehension and that simultaneity is always relative to the subject's apprehension and includes backwards and forwards relationships: "But it is not possible to cognize simultaneity [*Zugleichseyn*] except insofar as, beyond my action of grasping it together, I can apprehend (not merely think) the multiplicity as given both forwards and backwards. There must thus be given in perception an intuition in which the manifold is represented outside and beside each other [*ausser einander und neben einander*], i.e., the intuition which makes possible the representation of space [*Raumesvorstellung*]" (Kant, *Handschriftlicher Nachlaß–Metaphysik Zweiter Teil*, AA 18:616).

17 For Kant the hydrodynamic of matter in the universe depended on a quasi-static law (see De Bianchi, 2013).

18 The original text reads: “Alle Erscheinungen der Materie u.[nd] ihre Beweg.[enden] Kräfte sind mit dem ganzen Vnivers.[um] verbunden weil Raum u.[nd] Zeit absolute Einheiten sind. Man kann daher ein allgemeines Princip ihrer Wechselwirkung aufeinander annehmen was in realen Verhaltnissen zu einander besteht u.[nd] Erfahrung ist nicht anders möglich als in so fern jeder Gegenstand mit jedem andern in dieser Wechselwirkung gedacht und a priori in der Erscheinung gegeben angenommen wird.”

19 This is justified by Kant’s doctrine of representation (*Vorstellung*) and in particular by his definition of intuition as singular representation. This trait of uniqueness is proper of spatiotemporal intuition.

20 From this and other passages in the OP, it emerges that Kant’s view of a description of the beginning of the world confronted the Platonic account, such as that presented in the *Timaeus* where it is stated that time emerged together with the motion of the global sphere of fixed stars, see OP 21:33. It must be noticed that Plato’s *Timaeus* offers a ‘second’ perspective (or at least I interpret it in this sense) of the cosmogonic account. In the dialogue an alternative discourse, different from the one offered by the ordering procedure of the demiurgic agency, regards the *chôra*, the all-encompassing receptacle that is self-moved and moves, producing oscillations that generate imperfect configurations of the geometrical shapes and mathematical entities dictating the fundamental structure of matter. The affinity between this account and other approaches, including Kant’s, in portraying the universe as a systematic whole lies in the fact that it focuses on describing the collective behaviour of the parts, a trend of the dynamic (e.g., expansion, compression, etc.) rather than an exact sequence of ordering and causal relations.

Part II
Time and Timelessness in the Philosophy of Physics

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4. Time is Order

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Abstract
In this Chapter I argue that the fundamental aspect of our notion of time is that it defines an order relation, be it a total order relation between configurations of the world or just a partial order relation between events. This position is in contrast with a relationalist view popular in the quantum gravity literature, according to which it is just correlations between physical quantities that we observe and which capture every aspect of temporality in the world, at least according to general relativity. I argue that the view of time as defining an order relation is perfectly compatible with the way general relativity is applied, while the relationalist view has to face some challenges. This debate is important not only from the perspective of the metaphysics of space and time and for interpreting our physical theories, but also

for the development and the understanding of theories of quantum gravity.

4.1 Introduction

Time, or spacetime, is a fundamental ingredient of our physical theories. However, research in quantum gravity has been taken to suggest that time may not be fundamental and some authors, such as Rovelli (2011), go as far as to claim that one can dispense of time altogether. These claims are partially based on some relational views of classical physics in general and of general relativity in particular according to which all the physical content of classical theories is captured just by correlations between physical observables. In this Chapter, I argue against this view by offering an alternative characterization of time, which I will argue captures better the notion of time of classical theories and that it is able to accommodate some physical predictions that bare correlations cannot accommodate. This view is that time is fundamentally an order relation between events. I will start in Sect. 4.2 by reviewing the notion of time in non-relativistic physics and by arguing that time plays two roles there: it defines an absolute order relation between configurations of a system/the world and it has a metric aspect, i.e., it defines a duration.

I will briefly review relationalism as is standardly understood: as a position that denies or is skeptical about the metric aspect of time. I will introduce the family of models defined by Jacobi's action as exemplifying relationalists intuitions, and I will argue that this sort of relationalism preserves the order relation. Then, in Sect. 4.3 I will introduce radical relationalism as a more radical version of relationalism that would even deny the physical significance of the order structure of time. Several authors in the quantum gravity community seem to hold

this position, motivated by a particular analysis of the diffeomorphism invariance of general relativity and by the structures in several quantum gravity approaches. This position is applicable also to non-relativistic physics, where the symmetry of the Jacobi action mimics the symmetry of general relativity, and a similar analysis could be performed. For this and other reasons, radical relationalism also applies to non-relativistic physics. In Sect. 4.3, I argue against radical relationalism by insisting in my positive characterization of the temporal order structure of non-relativistic models: it is indispensable for both our understanding of these models and for the physical predictions we extract from them.

Next I will move to the case of general relativity in Sect. 4.4. I will argue that the temporal structure is much more complex and subtle in this theory, but that we can identify the same two roles of time. First, general relativity defines a proper time, which is the time an ideal clock would measure if it followed a certain trajectory in spacetime. As in the case of classical physics this metric aspect can receive some skeptical arguments or one can argue that it is relational as at the end of the day we use physical clocks made of matter for 'measuring' this aspect of time. Second, in general relativity we also find an order relation, which corresponds to the causal structure of the theory. As I will argue, it is more complicated than in the case of classical mechanics as in this case the order is a partial order, but it is still an essential part of a general relativistic model. I will argue that, similarly to the case of classical mechanics, correlations between observables may fail to capture this order relation, and hence that the radical relational views of time can be challenged on these grounds. Finally, I will briefly comment on how this position affects the foundations of quantum gravity in Sect. 4.5. First, I will argue that the relationalist positions claiming that the problem of time of some approaches to quantum gravity

can be overcome can be challenged in light of what I argue in this article. And second, if spacetime functionalism is the way of making sense of spacetimeless approaches to quantum gravity, it is precisely temporal order the essential function of spacetime to be recovered by functional analysis.

4.2 Time in Non-relativistic Physics

Most of our intuitions about time and temporality are captured by the models of classical theories. At the same time, the temporal structure of these models is less controversial, and for this reason I will start by reviewing the temporal structure of non-relativistic theories and I will argue that time plays two roles in these theories: it defines an order relation and it defines the duration of temporal intervals. I will be taking Newtonian mechanics as a paradigmatic example of non-relativistic theory, but the notion of time I will be discussing extends to many other domains in physics and science. I will also introduce relationalism as a position about time that denies the importance of the metric aspect of time, while keeping the order structure.

In Newtonian mechanics, at least in the way Newton formulated it, time and space are taken to be absolute: there is an absolute space in which bodies move and an absolute time flowing uniformly and independently of material systems. We can model Newtonian time as a real-valued parameter *t*. Each value t_0 identifies an instant in absolute time, a value t_1, $t_1 > t_0$ identifies a latter instant, and the difference $t_1 - t_0$ gives the duration of the interval between both instants, i.e., it measures how much time has elapsed. In this sense, we can see absolute time as playing two roles: it defines an order relation between instants and it also defines a duration for intervals. In this paper I will

argue that these two roles of time are still present, even if modified, in our best physical theories, including general relativity. A part of my claim that may be controversial is that speaking about an order relation may suggest that I am implying something like that there is a difference between past and future, that the present is in some way special, or that there is an arrow of time.[1] Here I do not want to make or deny any such claim, and I want to remain neutral with respect to those debates. By order relation I just mean that in non-relativistic physics there is a fact about what happens before and what happens after. For instance, Newtonian physics predicts the positions of Mars and Venus the next 10 times the Earth is at its perihelion. This prediction is that of an ordered set, and there are facts about the positions for the first time, for the second time, and so on. In Newtonian physics, the same set but with the order reversed would also be a valid model which would define an alternative order relation. Here I want to remain neutral about whether one of the two is privileged or about whether the two are just equivalent representations of the same physics, and I want to insist that time is directly related to the ordering of events.[2] In this sense, while one can take different positions about temporal asymmetry, the fact is that the absolute time parameter in Newtonian physics naturally defines a sequence of events.

Absolute time is a notion of time that is quite strong, and relationalism appeared as a philosophical position that advocated for a less charged notion of time. Relationalists argued that one can have a picture of the world in which time is relational, i.e., not absolute, while keeping all the empirical content of Newtonian physics. Relationalist arguments criticize absolute time on the grounds that it is unobservable. For instance, Leibniz argued that a world and a copy of it in which everything happened 1 second later would be indistinguishable, and he denied that they

could represent genuinely different possibilities. In this sense, Leibniz argued for a temporal relationalism according to which it does make sense to locate instants with respect to each other but not in absolute terms.

Contemporary relationalist positions go a step beyond, and they also deny the metric aspect of time.[3] The basic intuition for denying the metric aspect of time is that we never measure time directly. Instead, what we use for keeping track of time are physical clocks. One could argue that when we say that one year in Venus lasts around 225 days what we are really saying is that in the time that Venus completes an orbit around the Sun, planet Earth has spun 225 times around its axis. Alternatively, we could translate those 225 days into oscillations of our most regular atomic clock, or into some other process in our favorite physical clock. In this sense, what we measure is not the absolute duration of processes, but its duration relative to other physical processes. This motivates the relationalist to deny the metric aspect of time.

Let me introduce Jacobi's model of Newtonian systems as a class of models that capture well the intuition of the relationalists and which has a strong similarity with general relativity. I will skip the technical details here and focus on the conceptually interesting part of this kind of model. I refer the interested reader to (Barbour, 1994) and (Gryb, 2010) for complete introductions to this kind of model. The Jacobi's action for any Newtonian system with energy E is given by:

$$S[q] = \int d\tau \sqrt{T(q,\dot{q})(E - V(q))}\,, \tag{4.1}$$

where T and V represent the kinetic and potential energy of the system as functions of the degrees of freedom of the system q and their velocities. When one finds the trajectories that extremize this action what one finds are

trajectories $q(\tau)$ which are very similar to the trajectories $q(t)$ one would find making use of the standard Newtonian machinery. One can see that both $q(\tau)$ and $q(t)$ describe the same sequences of events, for instance they may describe the trajectory of a football after being kicked. However, the values of t and τ at each moment are (generally) different. Moreover, Jacobi's action admits infinitely many solutions to its equations of motion which are equivalent, i.e., they describe the same sequences of configurations, but expressed as different functions $q(\tau), q(\tau'), q(\tau'')$, etc. In this sense, while Newtonian dynamics describes how systems evolve with respect to absolute time t, the dynamics of Jacobi's action describes how systems evolve, but with respect to an arbitrary (monotonic) parameter τ. We can build a Jacobi model describing our solar system to illustrate the relationalist intuition in the example above.

The Newtonian model tells us that Venus describes an orbit around the Sun between time t_0 and time $t_1 = t_0 + 225$ days, and that in this time Earth spins 225 times around its axis. If we build a Jacobi model for describing the same system, it also predicts that there is an instant τ_0 and another τ_1 in between which Venus turns around the Sun and in between which Earth spins 225 times. What makes the Jacobi model different is that τ_0 and τ_1, or their differences, are not fixed. That is, while in every Newtonian model describing the system we necessarily have $t_1 - t_0 = 225$ days, in the Jacobi model the difference $\tau_1 - \tau_0$ could be any positive real number, as there is an infinite number of solutions of the equations of motion of this model which describe the same sequence of configurations of the system, and each of these solutions assigns different values to τ_0, τ_1 and the difference between them. That is, it could be 225, but it could also be 1, π, or 10^{100}.[4] Now, the relationalist argues that any model is equally good for describing what is happening in the solar system, no matter

if it says that $t_1 - t_0 = 225$ days or that $\tau_1 - \tau_0$ is 1, π, or 10^{100}. As models with different temporal parametrizations or different absolute durations are empirically indistinguishable, relationalists deny that absolute duration or the values of a temporal parametrization have any empirical meaning.

Notice that it is when we consider the whole universe that the argument takes its stronger form, while if we consider just part of the universe, then it seems harder to deny that time carries some physical information in its metric aspect. For instance, imagine that for describing the dynamics of the solar system we used two Jacobi models: one describing the motion of Venus and another describing the notion of the Earth. Each of them would describe what each of the systems does, basically to rotate around the Sun and to spin around their axes, and each of them would be using an arbitrary temporal parametrization.[5] This means that these two models together fail to capture the relations between the two motions, i.e., the fact that Venus completes an orbit in the time in which the Earth spins 225 around its axis. To capture these relations we need to use the Jacobi model including both systems or to restore Newtonian absolute time. If we add Newtonian time back to the picture, we can compare both motions and recover the facts about how they relate. In this sense, the metric aspect of time can be understood as encoding how the duration of processes of a system relates or would relate to the duration of processes of another system. The skeptical argument about the metric aspect of time can be applied to denying that there is an absolute sense of duration, but it cannot be applied to deny that the metric aspect of time has physical meaning when applied to a subsystem of the universe, as it certainly encodes relations (real or potential) with other physical systems.

Now, independently of whether we accept the relationalist argument or not, the crucial point for my argument in this article is that it does not affect the temporal order structure of our models. That is, even if we deny that the values of τ or t are relevant, it is still true that all the equivalent solutions $q(\tau)$ of a Jacobi model represent the same sequence of configurations of the system. The temporal order in each of these solutions is the same, and it allows us to predict the ordered set of 225 positions that Venus will take each time the Earth completes a turn around itself. Furthermore, we take the order in this set to be a physical prediction of our models: if we obtained the same set of 225 positions in a different order (turning around the Sun in the opposite direction for instance) that would be a wrong physical prediction.

In this sense, the relationalist positions based on the study of the Jacobi action deny the metric aspect of time but keep its order role. Jacobi models are fundamentally analogous to general relativistic models. The reason for this is that the symmetry of the Jacobi action is a one-dimensional version of the diffeomorphism symmetry group of general relativity. Different solutions of the equations of motion of the Jacobi action describe the same trajectories in configuration space, i.e., the mathematical space describing all the degrees of freedom of a system, but they disagree on the way they parametrize these trajectories. That is, they disagree on the values of τ they assign to each instant of the evolution. In this sense, instead of defining a unique $q(t)$, what Jacobi action defines is an equivalence class of solutions $q(\tau)$. One can transform from one $q(\tau)$ to any other $q(\tau')$ in the equivalence class by performing a temporal reparametrization $\tau \to \tau'$, which is just analogous to the way one can change spacetime coordinates (or perform an active diffeomorphism) $x^\mu \to x'^\mu$ in general relativity. The fact that any choice of temporal coordinate τ

is as valid as any other, supports the argument of the relationalist, but it is important to notice that the order relation is independent of this choice, as any element in the equivalence class describes the same sequence of events occurring in the same order. I will use this fact to argue that the symmetry of models like Jacobi's or general relativity cannot be used as an argument for denying the importance of the temporal order structure in our models.

4.3 Radical Relationalism in Non-relativistic Physics

Having introduced the basic notion of time in non-relativistic physics and the two functions that time plays in this kind of theory we can discuss radical relationalism as proposed or implied by part of the quantum gravity community. I will argue that the sort of relationalism that is being proposed in the literature, at least in its strongest reading, denies the order aspect of time, and reduces physics to be contained in correlations between observables. I will argue that this view of physics is problematic from the point of view of non-relativistic physics, as there are perfectly valid non-relativistic models that cannot be accounted for if we take this view of time. This will anticipate my objection in the next section, since leaving the order aspect of time out of physics will be in conflict also with the more flexible order of relativistic physics. I will consider proposals by Carlo Rovelli, Francesca Vidotto, and Daniele Oriti who defend this sort of radical relationalism.[6] They make claims like "The quantities predicted by the theory are values of some variables when other variables have given values" (Rovelli and Vidotto, 2022, 21) and by "the theory" here they mean any classical theory. I will take their position to be that a classical theory predicts just the values of some variables

when others take some other values and nothing beyond this.

Crucially for my argument, this leaves out the order in which these values are taken. This reading of the relationalism of the quantum gravity community has been suggested also by Thébault (2012, 2021) and I am unaware of any discussion by the community rejecting that this is their position. Even if in some passages there is some ambiguity in their claims,[7] and the importance that the temporal order relation may or not have is rarely if ever discussed, I believe that the most consistent reading of most of their work is that it denies the physical meaning of the temporal order relation if there is no 'observable' associated to it. Here I am not interested in arguing in detail the extent to which this position is endorsed, but in arguing that it is problematic at a classical level and that it has consequences in the construction and interpretation of quantum gravity. One of the features of this position that I will challenge is that clocks are necessary for making sense of physical theories.

Let me mention that the motivation for endorsing this sort of relationalism is partly an analysis of the reparametrization invariance of general relativity using some concepts of gauge theory, most relevantly the concept of observable. In my opinion, this analysis is flawed at several points as some authors have argued in the literature (Maudlin, 2002; Mozota Frauca, 2023; Pitts, 2014, 2017) and it makes the correlation view of physics problematic. Here I won't discuss the details of such an analysis of gauge, and I will just argue that the view emerging from that analysis doesn't accommodate some of our notions and intuitions about time and reparametrization invariant systems and that it can be challenged. In this section I will just discuss non-relativistic

systems and I will leave the case of general relativity for Sect. 4.4.2.

When it comes to discussing Newtonian systems, this view postulates that t, the Newtonian absolute time, is a quantity that can be measured by making use of clocks and that the basic correlations that one observes are correlations of the form $q(t)$, where q is some configuration variable. By considering t to be a measurable quantity, a partial observable in their terminology, the correlation view is able to recover all the physical content of Newtonian physics. That is, we are able to translate claims from the language of the correlations $q(t)$ to any other formulation of Newtonian physics in which time is framed not as an observable but as an order of events. The correlations view is problematic when the temporal structure of a model is *not* included in the set of quantities with which one builds correlations. This is well exemplified in the case of the Jacobi action of a Newtonian system as discussed above.[8] Let me take the example of a system of two harmonic oscillators.[9] Solutions of the equations of motion for such a model represent a trajectory in configuration space parametrized by an arbitrary parameter τ. These are functions $x(\tau), y(\tau)$ as represented in Fig. 4.1. As discussed above, there are several ways of interpreting this model that keep as a prediction the order of configurations that any two reparametrization equivalent solutions define. In the radical relationalist view, this information is not considered part of the model and it is just the set of pairs x, y along a trajectory, in no particular order, that is considered to constitute the physical content of the model. In (Colosi and Rovelli, 2003; Rovelli, 2004) a similar model[10] is built and it is argued that it is only the possible values of the position of one of the oscillator when the other takes a given value that has physical meaning, i.e., $x(y)$ or $y(x)$. Explicitly, they claim that one can build the

model using a 'clock' τ, but that this is just a gauge parameter that one needs to 'throw away.' That is, to eliminate time and keep $x(y)$ or $y(x)$ as the putative physical predictions of the model.

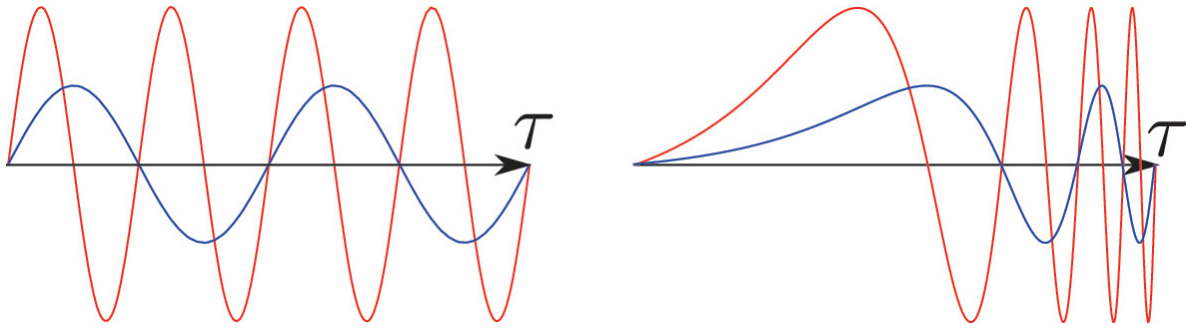

Fig. 4.1 Representation of two equivalent solutions of the equations of motion of the double oscillator model. The two solutions define the same sequence of configurations but disagree on the value of the time coordinate τ they assign to each configuration

There is nothing inconsistent with saying that the goal of building a model is to define a set of possible observations like x, y. However, when we use such models we extract more information from them than just which pairs x, y will obtain. As discussed above, we can also extract an order for the pairs x, y from τ, and even the Newtonian duration of intervals can be recovered as a preferred parametrization. In this sense, there are three possible interpretations of the model, depending on how much physical structure we read from it: we have the (not-ordered) set of pairs x, y, the ordered but arbitrarily parametrized trajectory $x(\tau), y(\tau)$ and the trajectory in Newtonian time $x(t)$, $y(t)$. Rovelli and Colosi invite us to prefer the first option, and it has been further argued that gauge invariance forces one to take it, but this is just a bad application of the concepts of gauge theory beyond its domain of applicability.[11] I will argue that the most natural interpretations of this type of model, and the most common ones, are the second and third, as they incorporate facts about order that we consider physical.

Imagine that we are using the double harmonic oscillator model to model precisely a set of two harmonic oscillators that we have in a lab and that we can observe and experiment with. Then, it is natural to ask what the position of one oscillator will be the next time the other one reaches its maximum amplitude. In this context we are not interested in the whole set of values of *x* that are taken when the other oscillator takes a position y_0, but just in the *next* one. Similarly, for the Venus example we are interested in the succession of positions that Venus will take each time the Earth spins, as it will be compared with the succession of observations we will make from Earth. From a practical point of view, we clearly take the order that the temporal structure of the model defines to be meaningful.

The radical relationalist could argue that the order relations that a model defines are meaningful only in as much they encode correlations with an external observer or measuring system. In this way, they would translate the claim that $x(\tau_1) = x_1, y(\tau_1) = y_1$ happens before $x(\tau_2) = x_2, y(\tau_2) = y_2$ into the claim that when the double harmonic oscillator couples with an external system with configurations represented by *w* the correlations x_1, y_1, w_1 and x_2, y_2, w_2 are allowed, with $w_2 > w_1$. The relationalist can try this kind of move, but in doing so they would be acknowledging that we many times model just a part of the degrees of freedom of the universe and they would have to acknowledge that it is not just what happens to a system that we model, but also when it happens, even if they believe that this *when* just encodes a relation with the rest of systems of the universe. This is in tension with the attitude held by the quantum gravity community, as they explicitly claim that they do not aim to make complete models of all the degrees of freedom of the universe, but just of part of it. This can be seen for instance in (Vidotto,

2017), when Vidotto claims that cosmology is different from 'totology,' the science that would aim to describe every single detail of the universe. If these authors agree that the model does not aim to cover every degree of freedom of the universe and that correlations with other systems can be encoded in the order relation, then they shouldn't dismiss the temporal order as completely unphysical.

An example where this clearly shows up is in the study of cosmological models. For many models one can leave the relativistic subtleties aside and consider models which are effectively non-relativistic and temporally reparametrization invariant. The most simple of them describes just one degree of freedom, the scale factor a which roughly gives the size of the universe, evolving with respect to an arbitrary parameter τ. One can read this sort of model as describing how a universe expands, contracts, or undergoes several expanding and contracting phases. There is even a way of defining a metric time, that corresponds to the proper time that some observers in the universe would measure. In this sense, this is a perfectly fine model from the perspective of the defenders of time as order, of the defenders of time as order plus duration, and also from the perspective of the cosmology community, that uses this model routinely in introductions to cosmology. Radical relationalists inspired by quantum gravity on the other hand, claim that in order for this sort of cosmological model to make sense, one needs to introduce some other degree of freedom in the model that would act as clock.[12] The basic reason for arguing for this supposed need is that you need at least two variables to speak about correlations, and if there is only one variable in the model one simply cannot build them. In this sense, the correlations view is of more limited application than the view that temporal order is an important feature of our physical models, at least of the non-relativistic ones.

As I said above, the relationalist could take the basic model with one degree of freedom to encode relations with external systems in the temporal order of the model, however, they do not do so and they claim that it is necessary to explicitly add other degrees of freedom to the system in order to make sense of it. In this sense, it seems to me that they are rejecting a plausible interpretation and a respectable model and getting one step closer to the 'totology' they wanted to avoid.

I have argued that from an external perspective, temporal relations can be considered physical and meaningful in a sense that does not seem in contradiction with the views of the relationalist if we interpret them as encoding relations with another system or observer. This already supposes a challenge for the relationalist. But one can also challenge the relationalist from an internal perspective. That is, we can build a model that represents a whole universe with nothing external to it or with no degree of freedom left out of the model and still claim that the temporal order of the model is meaningful. For instance, one can consider that the cosmological model describing how a varies is perfectly fine for describing an empty universe with its size fluctuating or that the model of the double harmonic oscillator represents a universe containing just two particles oscillating. For starters, these models and interpretations seem perfectly fine and consistent, and the relationalist would need to provide some good reason for not considering them as physical possibilities.

The arguments by the relationalists are very tightly connected with epistemology and with the way observers perform observations. In this sense, the empty universe model is considered physical only when another system is included in the model to 'measure' how the universe expands and contracts. For someone with realist

inclinations, rejecting this model for the case in which no other degree of freedom may sound like rejecting that a tree falling in a forest with no one around makes a sound. Epistemological consideration can constrain our metaphysics of science, but one can suspect that the relationalist has taken it too far. Similarly, for models including human beings one can also raise skeptical arguments questioning the order relation. For instance, given my memories and records of yesterday, I can hold the belief that yesterday was real, that it happened before today, and that it had an influence on the events and states of affairs of today. But if the world was just an unordered set of things that happen, as the relationalist seems to suggest, or if they happened in an arbitrary order, I would also have access to the same state of the world as it is now, and I would also hold the same beliefs about yesterday. However, one could push this argument a bit further and raise the question of whether the instant of yesterday was or is in some sense real. The relationalist may want to deny that it happened before today, but they still want to claim that it happened or happens. In this sense, one has to be careful with skeptical arguments, as they can push one too much into denying aspects of our physical or metaphysical theories that we may not want to deny. One can draw an analogy with properties like mass or charge. An argument can be formulated for denying that they are physical properties that we can measure, as one can argue that what we measure is the position of particles or bodies and that from their motions we infer their properties. One can reasonably believe that even if these properties are not directly measurable, they play a role in our understanding of the world and that we should consider them to be physically relevant. Similarly, even if one does not have a way of directly measuring temporal order, it can be argued that it plays a role in our theories and worldview and one can resist skeptical claims denying its importance.

If the relationalist pressed on this point and asked what role the order structure of time plays in our theories and worldview, one naturally would think in terms like evolution, causality, or locality. None of them is uncontroversial, and I do not want to enter into the details of the arguments concerning them, but it seems that our (non-relativistic) theories relate what happens at one instant with what happens immediately after and immediately before by means of the equations of motion of our theories. Of course, the equations of motion also relate temporally distant instants, but the influence of one instant can be seen as being continuously carried from one instant to another, and the form of the equations of motion suggests this. Similarly, the notion of physical influences propagating locally in space but also in time is ubiquitous in physics, and this supports the claim that the order structure of time plays a crucial role in our understanding of the world. Notice also that relationalists like Rovelli make use of the same equations of motion relying on the ordering structure for deriving their predictions in the form of correlations, even if they later on throw the order structure away. One could suspect that when they are doing so they are also throwing away important information about the world that allowed them to make the predictions in the first place.

Let me conclude this section by insisting that radical relationalism that throws away the temporal order structure of non-relativistic models may be throwing away physically relevant information about the system when doing so, and that, for the case in which the model is considered externally, I believe they would be forced to accept it, as the order structure of the model is related with facts about how it interacts with other systems. In the next section I will expand this conclusion to the case of general relativity, which has a more complex temporal structure but similar reasonings will apply.

4.4 Time in General Relativistic Physics

At first sight, the discussion above may seem irrelevant to a defender of the correlation view. After all, their view is motivated by general relativity and quantum gravity and I have been discussing non-relativistic models. However, I will argue that the arguments above can easily be extended to the case of general relativistic case. In the following, I will argue that in general relativity both roles of time, the order and the metric roles, are present, and that as in the non-relativistic case one can raise skeptical arguments about the metric role, but that the order role seems more fundamental. I will then express the explicit criticism of the correlation view as applied to general relativity.

4.4.1 Temporal Order Relation in General Relativity

In relativity theory (both special and general), the split between absolute space and absolute time of Newtonian physics is broken and instead one has spacetime. In general relativity, spacetime is dynamical in the sense that depending on the configuration of matter (and even this does not fix spacetime) one can have different spacetimes. However, I will argue that even if spacetime is a more sophisticated structure than just space and time, it still hosts the two roles of time in non-relativistic physics.

The notion of simultaneity in relativity does not play a role, and even if one can divide spacetime into space and time, there are many ways of doing so, and none seems to be physically significant in any meaningful way. In this sense, one doesn’t speak about instants in relativistic physics but of events or spacetime points. Even if in relativity theory there is no physical structure defining whether two points are simultaneous, the theory

incorporates an order structure that allows us to say that certain events happen before or after other events. This is the light-cone or causal structure of the theory, which distinguishes between space-like, time-like, and light-like pairs of points. When a pair of points is time-like or light-like one can say that one happens before another, and this defines a partial order relation. That is, the causal structure of the model tells us for any two points p and q whether p is in the past of q, in its future, or neither.[13] Bodies and particles in spacetime follow non-spacelike trajectories, i.e., we can see the history of every body as going from its past to its future as defined by the light-cone structure of the theory.

Similarly, the laws of relativistic physics respect this causal structure: charged particles generate an electromagnetic field in the points in its future light cone, to predict the state of the world in a region one needs to know what happened in its past, and so on. As happened in the case of non-relativistic physics, for many physical applications this causal order may be associated with something like an arrow of time or a future-past asymmetry. In this article I want to remain neutral about this possibility, as there is nothing in the formalism of special or general relativity that indicates a preference about this issue. For any spacetime, there are two possible orientations one can choose for the partial order relation, and one could argue that only one is right and that physics should reflect this asymmetry. As in the case of non-relativistic physics, my claim that there is a partial order relation is compatible with views that accept and that deny some sort of temporal asymmetry.

The metric aspect of time is also present in relativistic physics. Given any time-like trajectory in spacetime, relativity defines a time that parametrizes the trajectory, the proper time, that is interpreted as the time an ideal

clock would measure if it followed that trajectory. This time is trajectory dependent, which means that two observers following two different trajectories in spacetime that meet at two points will measure different times between both events. This gives rise to the famous twins' paradox and other relativistic effects, it has been empirically corroborated, and it is an effect that is relevant for the correct working of devices like the GPS. As in the case of non-relativistic physics, one can formulate skeptical arguments for denying the physical relevance of this metric aspect. Proper time is defined as the time an ideal clock would measure, but we do not have ideal clocks, what we have are just physical clocks. In this sense, if our theory was able to incorporate all the degrees of freedom of the universe, including the ones that constitute our clocks, why would we need this time that we cannot measure? As in the non-relativistic case, retaining what happens in spacetime and preserving the non-metric structure of spacetime may be enough to make sense of relativistic models. We can even bring the example of the Earth and Venus to a relativistic setting: in a relativistic model of the solar system we can also make claims like that in the time Venus does an orbit around the Sun, Earth spins 225 times around its axis.

The analogy with the non-relativistic system also forces us to the conclusion that one could deny the physical relevance of the metric of time only when all the degrees of freedom of the universe are considered. When taking the perspective that some degrees of freedom are left out, proper time can be seen as encoding physically meaningful predictions for systems related to spacetime or the material systems considered by the relativistic model. For instance, even if a relativistic model describing a spacetime does not describe or take into consideration facts about a space rocket traveling in spacetime, it can be used for computing its proper time, which would be a good approximation to

the readings of the clocks in the rocket. In this sense, relativistic models are not that different from Newtonian models. Time still plays an ordering role, as one can still make claims like that the event of me having breakfast tomorrow is in my future and that the Big Bang singularity was in my past. Time still plays a metric role, as there is some sense in which one can say that from the big bang to the moment in which I am typing this 13.8 billion years have elapsed and that from this moment to tomorrow's breakfast my watch will advance a few hours. Things are slightly more complicated, as now we have to accept that simultaneity is lost, that the readings of clocks depend on their trajectories, and that the order structure is not independent of what matter does. But we still have a temporal structure playing similar roles.

I believe that from a conceptual point of view one cannot deny that both elements, order and metric, are present in relativistic models. However, it is when one discusses the diffeomorphism invariance of general relativity that confusion is prone to arise. General relativistic models are defined making use of the structures of differential geometry as a triple $\langle M, g, \phi \rangle$, i.e., a manifold M, a metric tensor g defined on it and the matter content of the universe, represented by ϕ . Given one triple, we can identify events with points on the manifold and the metric tensor defines the temporal relations between them. For instance, the event of me typing this can be represented by a point p and the event of tomorrow's breakfast by a point q. My trajectory in spacetime is represented as a trajectory on M and the metric tensor on the manifold determines that this trajectory is timelike, i.e., that breakfast is in my future and its duration of some hours. The diffeomorphism invariance of the theory means that one can define an equivalent triple $\langle M, g', \phi' \rangle$ in which p and q do not represent the same events, as it is p' and q' which

represent me typing and my breakfast tomorrow. However, using this alternative model one can also read that my breakfast is a few hours in my future by reading the information encoded by g' in the set of points that represent my trajectory in this model. In this sense, any diffeomorphism-related triple $\langle M, g, \phi \rangle$ contains the same physical information and one can claim that it is the equivalence class of triples under diffeomorphisms that really constitutes the relativistic model. This symmetry of the theory is what motivates relationalist claims like the ones I have introduced in the previous section.

Relativistic models are analogous to the Jacobi model for Newtonian systems. Recall that for the double harmonic oscillator system we claimed that all parametrizations of the same trajectory in configuration space represented the same physics. This can be formulated as saying that it is in the equivalence class of trajectories $x(\tau), y(\tau)$ that one has to extract the physics of the model. As I argued above, all the trajectories agreed on assigning the same order to the configurations that the system takes, and it is reasonable to take it to be part of the physical content of the model. For the case of relativistic physics exactly the same reasoning applies: every triple $\langle M, g, \phi \rangle$ in the equivalence class represents the same events with the same partial order relations holding between them, and one can consider them to be physical. That is, it is a prediction of a general relativistic model describing the universe surrounding me not just that I am typing and that I will have breakfast, but also that the latter is in the future of the former, and that my clock will advance a few hours in between both moments. To conclude this Subsection, let me insist on its main message: in relativistic physics the temporal structure is more sophisticated and it is even dynamical in the case of general relativity, but we find that it plays the same two roles as it still defines a (partial) order relation between

events and it predicts the readings of clocks. The fact that the models are expressed using manifolds and coordinates does not change the fact that these temporal functions are well-defined. In the next subsection I turn to analyze how the radical relationalist interpretation is applied to general relativity and the way it is still affected by my arguments in Sect. 4.3.

4.4.2 Radical Relationalism in General Relativity

As I commented in Sect. 4.3, the radical relationalism that is advocated for by the quantum gravity community may be too radical and throw away part of the content of our physical models that we may regard as physical. In this section I will argue that the same arguments expand to the way radical relationalism is applied to general relativity.

First, I noticed that when discussing what observables count as legitimate observables, Rovelli and Vidotto took that the Newtonian time counts as an acceptable observable in the context of Newtonian dynamics. Why shouldn't we extend this to the case of general relativity? Why shouldn't we incorporate the temporal structure of the model as an observable? The temporal structure of general relativity is more complex than the one of Newtonian mechanics, but both seem equally observable or unobservable. In the same way that Rovelli and Vidotto accept that t can be considered an observable in Newtonian mechanics because one can measure it with a clock, why not accept τ, the proper time along some time-like curve to be an observable for the same reason?[14]

A similar comment could be made about coordinates. There are spacetimes for which one can undoubtedly construct systems of coordinates that are closely related to the temporal structure of the spacetime. The clearest example is Minkowski spacetime, where the standard

coordinates can be interpreted as the proper times that some ideal clocks would measure in exchanging light pulses among them. Taking this coordinatization seems physically meaningful, and there seems that there is no reason not to take it as a partial observable, if we accepted that in a Newtonian theory t could be taken as a partial observable. If one accepts one set of coordinates as meaningful and observable, one could even argue that any other set of coordinates should be considered on the same footing, as any function of a set of observables could also be considered equally observable. For more general spacetimes, engineering a way of associating spacetime coordinates to something like ideal clocks can be more complicated, but there are ways of doing so, at least for some spacetime regions.

An ingenious example of this was developed precisely in Rovelli (2002a). This idea required four clocks moving in spacetime and sending signals into space indicating the proper time at which they emitted the signal. From any point in at least some region of spacetime one receives the four signals and this defines a coordinatization for spacetime. Rovelli baptized these coordinates as the 'GPS observables' and claimed they are legitimate partial observables for the model when these four clocks and the signals they send are included in the model. But why would we limit this to just the case in which we have explicitly added the clocks and signals to the system? This brings us back to my objection in Sect. 4.3 that radical relationalism goes against the standard practice of physics in the sense that models are many times used in a way that encodes relations of a system with some others external to it. Even if a set of clocks filling spacetime with signals is not added to the model, one can see a model written in terms of GPS coordinates, or any other coordinate system, as encoding the same physics as the model in which we explicitly add

such clocks. In this sense, even if we do not add our worldline to a solar system model or to a cosmological model, a general relativistic model predicts the succession of events that we will go through and the time that our clocks will indicate. Similarly, radical relationalism in general relativity has motivated claims like that we should include clocks and rods as internal degrees of freedom of our theories.[15] In this sense, it rejects the idea that a prediction for a physical quantity O at a spacetime point, $O(x^\mu)$, is meaningful if it is expressed in terms of coordinates in the manifold, and it considers it meaningful only if it is expressed as a correlation with four physical quantities that would be acting like rods and clocks, i.e., individuating spacetime points. That is, a correlation $O(\phi^a)$ would be considered a physical prediction, where ϕ^a represents four physical degrees of freedom such as four scalar fields.

In the same way that in the case of non-relativistic physics I considered this view to be problematic, I believe my analysis extends to the relativistic case. Indeed, in the usual scientific practice one rarely adds explicitly any clocks and rods to the model, and we are able to make sense of vacuum models where no matter is added. Vacuum models in general relativity present a rich phenomenology and are widely used for describing gravitational waves, cosmological settings, or the merge of black holes among others. From this perspective, the 'external' view of general relativistic models seems valid, and the temporal partial order structure of the model can be seen as encoding physical information that observers not explicitly added to the model would observe. From the internal perspective one can also raise a challenge to the radical relationalist, as one can still believe that a model describing, say, the merge of two black holes, is a physically meaningful model, even if no matter is included in the model.[16]

Besides these arguments, in Sect. 4.3 I argued that the temporal order of events plays an important role in grounding many of our physical intuitions like the ones related to locality, evolution, or causality. Moreover, the differential equations describing our non-relativistic physics explicitly relate the state of the world or of a system at an instant with the state of the world in the instants just before and after. This sort of argument can be extended to the case of relativistic physics, as in relativistic physics it is generally the case that locality, evolution, or causality play an important role in our reasoning and one still has local differential equations that relate the state of affairs in each spacetime point with what happens in the points surrounding it. In this sense, the order structure plays an important role in our understanding of relativistic theories.

Finally, it is clear that in relativistic models one does not just take the empirical content of a model to be contained in a non-ordered set of multiplets ϕ, g corresponding to the state of the matter fields ϕ and (possibly) of the metric field g at every spacetime point, as the order that the metric tensor, together with the manifold structure, defines is taken to be part of the content of the model. This is just as in the double harmonic oscillator model, where one takes the ordered set of pairs x, y to carry physical content, and a differently ordered set wouldn't be considered the same physical possibility. In the case of the relativistic model if we assigned each combination ϕ, g to a random point on a manifold we could equally claim that we have built a different (and physically wrong) model. For these reasons, I believe that the claims in the quantum gravity literature that in relativistic physics it is just correlations between observables that have physical meaning are misguided and that they leave out an important part of the physical content of our relativistic models. Time, in

relativistic physics, is (at least) a partial order relation between events, and any account that leaves out this feature seems to be leaving out some important physical content of our models.

4.5 Time and Quantum Gravity

In this last section I want to make a brief comment about the implications that the analysis above has for two relevant issues in the foundations and philosophy of quantum gravity.

First, in canonical approaches it is well-known that there is a problem of time: by applying canonical quantization methods to reparametrization invariant models one gets a quantum model with a trivial evolution equation. This has been interpreted as showing that time, as the (partial or not) order relation of the original model, does not play a role in the quantum theory. Radical relationalists argue that this is not problematic because the quantization procedure still leaves room for building some quantum operators that look like correlations, which from the radical relationalist perspective is all you need to build a physical theory. For instance, for the double harmonic oscillator case there would be operators associated with x and y, and the hope would be to build operators $\hat{X}_Y$ that would capture the correlations between both observables. If one accepts the analysis above, one would very reasonably worry that some important physical content of the classical models (both for Jacobi models and general relativity) has gone missing and that there is something problematic about the quantization carried out. In other words, they would take the problem of time to be something serious and be skeptical about the way the radical relationalist pretends to solve the problem or argues that the problem is

not a problem after all. For an analysis of the problem of time along these lines, see (Mozota Frauca, 2023).

Second, one may nevertheless have an open mind about the possibility of a fundamental quantum theory of gravity in which there is no time at a fundamental level. In this case, it seems that something like spacetime functionalism, as discussed in the philosophy of physics literature (Lam and Wüthrich, 2018, 2021) would be needed for recovering the appearance of a spatiotemporal world. This amounts to saying that in order for a spacetimeless theory to be compatible with our experience, it needs to have structures able to satisfy the functions that we usually ascribe to space and time. Following the analysis here, the main function of time that would need to be recovered in order for a functionalist account to be successful is precisely the order function. In this sense, it seems reasonable to demand an explanation from the spacetime functionalist or from the quantum gravity theoretician about how the order function of time can be recovered from a fundamental level of reality that would be lacking it.

4.6 Conclusions

In this Chapter, I have argued that time plays two major roles in our physical theories: it defines an (partial or total) order relation and it has the metric role of defining the duration of processes. I have argued that while one can raise some skeptical arguments about durations, the order role is fundamental and that it seems essential for our physical theories. In this sense, I have argued that some radical relationalist views by the quantum gravity community are problematic and I have commented that this has an impact on our understanding of the philosophy and foundations of quantum gravity. I believe that for the picture of a fundamentally spacetimeless world that nevertheless looks spatiotemporal to be appealing, the

quantum gravity community should address the issues raised in this article and explain how the appearance of our world as a partially ordered set of events could be compatible with their proposals.

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Footnotes

1 Similarly, the order relation of time can be connected with other philosophical issues such as our perception of time our the notion of causality. I will also take a neutral stance with respect to those topics.

2 Some authors prefer referring to the ordering structure of time as defining a ‘betweeness’ relation rather than an order relation in order to make clear that there is no privileged temporal direction in the formalism.

3 This sort of view was famously defended by Ernst Mach (Mach, 1883), who inspired Albert Einstein in his formulation of general relativity. More recently,

and in relation to the debates surrounding quantum gravity, this view has been defended by Julian Barbour (Barbour, 1994).

4 Perhaps the reader has noticed that while in the Newtonian model t carries a unit (days), for τ I am dropping it, as it is just an arbitrary parameter.

5 I am at this level of discussion ignoring the gravitational interaction between both systems.

6 For specific details, see (Marchetti and Oriti, 2021; Oriti, 2021, 2023; Rovelli, 1991, 1996, 2002b, 2004, 2011; Rovelli and Vidotto, 2015, 2022; Vidotto, 2017).

7 For instance, in (Rovelli, 2004) it is claimed both that the arbitrary coordinates have no physical significance (Rovelli, 2004, 79) and that there are sequences of events (Rovelli, 2004, 74).

8 A terminological note: Rovelli and Vidotto call this sort of system relativistic as they interpret it as a model with no preferred time variable. I prefer to stick to the traditional terminology, which in my opinion is more adequate as this model is not a model of special or general relativity, and consider this a non-relativistic model.

9 This example has been studied in (Mozota Frauca, 2023) also in connection with the problem of time of quantum gravity.

10 Colosi and Rovelli fix the frequencies of both oscillators to match, which highly simplifies the trajectory in configuration space.

11 I refer the reader again to (Maudlin, 2002; Mozota Frauca, 2023; Pitts, 2014, 2017) for discussions of why the concepts of gauge theory are not readily applicable to general covariant models.

12 This is explicit in (Vidotto, 2017, Sect. 15.3.2) and (Marchetti and Oriti, 2021).

13 For the case of spacetimes with closed timelike curves a point p can be both in the past and in the future of another q, I won't be explicitly dealing with this sort of spacetimes, although the local order relation is perfectly fine and unproblematic. The definition of causal structure I am using assumes that spacetimes are time-orientable, which is a widely held requirement. For a discussion of some of these subtleties I refer the reader to (Manchak, 2011).

14 In (Rovelli, 2004, 51–52) it is discussed a solar-system example in which proper time is included as an observable but just when a clock is explicitly added to the system. Here I am proposing to include the proper time along any time-like curve to be considered an (partial) observable of the theory.

15 See for instance the recent discussion in Marchetti and Oriti (2021).

16 Let me mention that the relationalist could try to argue that radical relationalism can account for such situations too by studying correlations between gravitational observables. Let me make three comments about this line of reasoning. First, this is not a strategy followed by the quantum gravity community at the time of interpreting their models. Second, as far as I know, it seems impossible to build coordinates out of the metric and related tensors. Third, for symmetric cases like Minkowski spacetime one has spacetime points with the same physical properties, and hence one cannot build coordinates based on the metric that would distinguish them.

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5. *En Route* to Reduction: Lorentzian Manifolds and Causal Sets

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Abstract
This Chapter presents aspects of causal set theory (a research programme in quantum gravity) as being *en route* to achieving a reduction of Lorentzian geometry to causal sets. The Chapter takes reduction in philosophers' sense and argues that the prospects are good for there being a reduction of the type envisaged by Nagel. There is also discussion of the prospects for the stronger functionalist variant of Nagelian reduction, that was formulated by Lewis. One main theme will be causal set theory's use of a physical scale (viz. the Planck scale) to formulate how it recovers a Lorentzian manifold. This use illustrates various philosophical topics relevant to reduction, such as limiting relations between theories, and the role of analogy. The Chapter also emphasises causal set theory's probabilistic method, viz. Poisson sprinkling: which is used both for formulating the reduction and for exploring its prospects.

5.1 Introduction: Reducing Geometry

The aim of this Chapter is to present aspects of causal set theory, which is a research programme in quantum gravity, as being *en route* to achieving a reduction of Lorentzian geometry to causal sets: a reduction in philosophers' sense that a theory T_t is shown to be part of, i.e. encompassed by, another theory T_b.[1] Agreed, the fact that causal set theory aims for such a reduction is not news. That much is clear from a brief perusal of the causal set literature, despite differences in the jargons of physicists and philosophers.

Indeed, it is clear from the paper that launched the programme (Bombelli et al., 1987). For it gives a detailed proposal about in what sense a Lorentzian manifold approximates a causal set: or in converse jargon (which I will also use), in what sense a causal set recovers a Lorentzian manifold.[2] But as a case-study in reduction, causal set theory has many features that make it merit a more detailed examination by philosophers. Most, though not all, of these features relate to two current themes in philosophy of physics: (i) the contrast between reduction and emergence, and (ii) spacetime functionalism. So I will introduce these features by recalling these themes: in Sects. 5.2 and 5.3 respectively. As we will see, the themes are related: for one version of functionalism is *functionalist reduction*. Causal sets are discrete: so the proposal that a Lorentzian manifold approximates a causal set raises the general topic of what we should mean by a continuous structure approximating a discrete one. So Sect. 5.4 addresses this in general terms. Then Sect. 5.5 reports how Sect. 5.4's ideas play out in causal set theory's proposal.

These Sections and this Chapter as a whole have a limited scope, as regards both physics and philosophy. As to physics, I will set aside causal set theory's treatment of matter and radiation (which involves "decorating" causal

sets with appropriate fields), and the theory's proposed treatment of quantum theory (which emphasises path integrals). So I focus on just the relation between continuous and discrete conceptions of spacetime, taken as both vacuum and non-quantum. I will also downplay dynamics: indeed, in two senses. First, I set aside the effort to define a class of causal sets appropriate for recovering Lorentzian manifolds by formulating a rule for generating them over time. Second, I set aside the effort to recover the Einstein field equations (the holy grail of general relativity); i.e., for a vacuum spacetime, the Ricci tensor being zero. For all these topics in physics, it must here suffice to say that causal set theory has since 1987 delivered plenty of positive results; though there remains, of course, plenty still to do.[3]

This Chapter also has a limited *philosophical* scope, in two main ways. First, I will set aside controversies about how to conceive of physical theories. In the last fifty years, the main debate has been whether we should take a theory to be (i) a set of sentences that is closed under deduction (called 'the syntactic conception') or (ii) a set of models, where a model is, roughly speaking, a solution or approximate solution to the theory, taken not as a linguistic object but as a model in the sense of model theory and formal semantics (called 'the semantic conception'). But I concur with recent critiques of this debate, urging that the contrast has been overplayed. In brief, I think critics of the syntactic conception (a) ignored the fact that it of course took the sentences to be interpreted, and almost always to be in a natural language, not a formal one, and (b) neglected the fact that our only access to models is through (interpreted!) language, so that the semantic conception is after all closer to the syntactic one (see, for instance, Lutz (2017)).[4]

Second, since space is short: I will not comment in detail on recent philosophical discussion of causal set theory, which has indeed emphasised the themes of the reduction of Lorentzian geometry, and of spacetime functionalism; especially two papers by Lam and Wüthrich (2018, 2021) and Huggett and Wüthrich (forthcoming) discussions in Sects. 2.4 and 3.3. My reason, in brief, is twofold. First: I for the most part agree (especially with the discussions in the later book); and where their expositions complement mine, I simply cite them. Second: my main disagreement can wait till Sect. 5.8.2. We will see that despite this limited scope, there is plenty to say: both by way of reporting causal set theory's results so far, and relating these results to philosophical themes.

Broadly speaking, Sects. 5.5 to 5.7 report results about the progress so far in reducing Lorentzian geometry to causal sets, with an eye on the more general themes of Sects. 5.2 to 5.4. After I recall the basics of causal sets (Sect. 5.5.1), Sect. 5.5.2 reports a construction for rigorously obtaining a Lorentzian manifold as the continuum (ultra-violet) limit of a suitably chosen sequence of causal sets that are spatiotemporally located ever more densely. But as I emphasise (already in Sect. 5.2.1): for causal set theorists, the physical interest of a causal set recovering a Lorentzian manifold lies in the regime *before* the continuum limit. More precisely, the interest lies in the regime of length-scales at and above the Planck length. This raises the question (both technical and conceptual) how we should make precise the idea of a discrete structure recovering a continuum, only at and above a certain length-scale (and thus differently than in Sect. 5.5.2). Though this question is addressed in Sects. 5.4 and 5.5 using broadly geometric tools, no agreed proposal emerges. A bit more precisely, using the jargon of causal set theory: what has come to be called the

Hauptvermutung, i.e., 'the main conjecture,' is yet to be formulated precisely, and proven. But as I go on to discuss in Sects. 5.6 and 5.7, there is plenty of favourable evidence.

So the '*en route*' in my title is a *double entendre*: it signals both that the physical interest lies in the regime before the continuum limit, and that causal set theory's reduction of Lorentzian manifold is work in progress. I will concentrate on one important theme—the use of probability as an investigative tool. Thus causal set theory has explored which causal sets are appropriate for recovering Lorentzian manifolds by selecting points from such a manifold randomly, i.e., by a certain probabilistic process, and defining a causal set on the selected points by endowing them with the causal relations to each other that they had within the manifold, i.e., "before" they were selected. This process assumes equiprobability for equal spatiotemporal volumes; more specifically, it is a Poisson process. So it is called *Poisson sprinkling*. But I think that since the causal structure is inherited from the manifold, a better name would be 'Poisson *extracting*' or 'Poisson *harvesting*.'[5]

This idea of Poisson sprinkling is "turned around" by another strategy for addressing the question, i.e., for formulating the *Hauptvermutung*, that causal set theorists have adopted since 2000. This strategy is reported in Sect. 5.6. In short, instead of probability being a tool for probing geometry, one uses probability to define a geometric notion. More precisely: the idea (Bombelli, 2000) is to define a distance between Lorentzian manifolds in terms of the *statistical angle* (in probabilists' usual sense of that phrase) between the probability distributions that they each define over the causal sets that can be obtained from each geometry, by Poisson sprinkling. This notion of distance turns out to have many merits.[6] Furthermore, Poisson sprinkling turns any function of sprinkled causal

sets (or as I prefer: harvested causal sets) into a *random variable*. For the causal sets that can be obtained by sprinkling from a given Lorentzian manifold[7] form a probability space, with its distribution defined by the sprinkling. Thus Sect. 5.7 reports how this opens up the prospect of showing that such a function, that encodes a discrete analogue of a feature of continuum geometry that we want to recover (such as manifold dimension, or the length of a timelike geodesic), has statistical properties that make it a good *estimator* of that continuum feature. For example, we can hope to show good behaviour of its mean and of its variance. As we shall see, causal set theorists have several such success stories.

The concluding Sect. 5.8 returns us to the philosophical themes of reduction, emergence and spacetime functionalism, presented in Sects. 5.2 and 5.3. It uses the details about causal sets, in Sects. 5.4 to 5.7, to assess the prospects for a reduction (perhaps a functionalist reduction) of Lorentzian manifolds to causal sets.

5.2 Reduction and Emergence

In general philosophy of science, it is usual to take reduction and emergence as contraries, i.e., as excluding each other. The reason is that while in reduction, T_t is a part of T_b, and so "says nothing additional" to T_b, in emergence, the emergent theory in some sense "says something additional." Although it is suitably linked to the theory from which it emerges, it has some kind of autonomy, or "a life of its own." Of course, authors vary about how to make these notions precise. In this Chapter, I will focus on the traditional idea of reduction, largely due to Nagel (1961, 1979). He adopts the syntactic conception of a theory as a set of sentences closed under deduction, and so conceives reduction as deduction, i.e., as a matter of

T_t being deduced from T_b.[8] The idea, simply put, is that T_b not only (i) implies T_t's predictions about observable states of affairs (within error-bars at least as good as T_t's), but also (ii) implies T_t's statements about unobservable states of affairs.

For doing only (i) would be compatible with T_b's being a rival or replacement of T_t, rather than a reduction of it.[9] For the moment, we only need to add to this sketch of reduction as deduction, two main qualifications: qualifications that are agreed by all hands, including Nagel.

(1) *Bridge laws*: Usually, T_t has concepts unmentioned (in linguistic terms: predicates unused) by T_b; so that for the deduction to succeed, statements linking these to T_b's concepts or predicates must be added to T_b. So it will be T_b *together with these statements* that jointly give the implications (i) and (ii) above. These statements, called *bridge laws*, can be (but might not be) *definitions*, in logicians' sense of a statement that determines the extension (i.e., set of instances) of the concept or predicate concerned. This will be a topic in Sect. 5.3.

(2) *Allowing analogy*: Often, what is deducible from T_b is not exactly the given (maybe: historical predecessor) T_t, but a theory T_t^* that is analogous to it. Though there can surely be no general account of what counts as a strong enough analogy to justify the name 'reduction' (as against calling T_t^* a rival or replacement of T_t), one surely wants the obvious revisions of (i) and (ii) above; as follows.

(i'): The deduced T_t^* implies T_t's predictions about observable states of affairs (within error-bars at least as good as T_t's); and

(ii'): T_t^* implies appropriate analogues of T_t's statements about unobservable states of affairs.

A standard example is the reduction of Galileo's law of free fall (implying that a falling body has the same acceleration at different heights) by Newton's theory of

gravity. For the latter implies that the acceleration decreases with height—but so slightly as to be unobservable with bodies dropped from usual heights, e.g. the Tower of Pisa; (condition (i')). And the concepts and vocabulary of Newton's theory, for both the observable and the unobservable, such as mass, inertia and force, are sufficiently analogous to Galileo's,[10] that condition (ii') is satisfied.[11] We will see this allowance of an analogous theory T_t^* in our discussion of causal set theory's recovery of Lorentzian geometry: already in Sect. 5.2.1 and throughout the Chapter, culminating in Sect. 5.8.

So much by way of sketching the notion of reduction.[12] *Prima facie*, it seems opposed to emergence, even allowing for (2). For one expects an emergent theory to have more of a life of its own than is compatible with being an analogue of some consequences of a putative reducing theory.

But in recent philosophy of physics, several authors make reduction and emergence precise in such a way that they are compatible.[13] In this reconciliation, one common idea has been that:

(A): for some physical theories, novel mathematical and conceptual structures (and associated phenomena) can be proven to appear, at the limit of some parameter; and

(B): in such cases, the theory obtained in the limit should be classified as emergent from (since so different from) the theory before the limit, and yet also as reduced to it (since rigorously derived from it using the limiting operation).

This is indeed my own view. Well-known examples in thermal physics include (a) the number of microscopic constituents of e.g. a gas sample, going to infinity, and (b) the length of the lattice-spacing in a lattice-system going to zero.[14] This view also accords with physicists' usage in their discussions of how any theory of quantum gravity is

expected to reproduce the success, within appropriate regimes, of our current well-confirmed theories of quantum fields defined on a continuum spacetime. Thus they usually talk of 'the emergence of classical spacetime from quantum gravity,' while clearly intending this to be also a case of reduction in philosophers' sense. Oriti (2023) is a fine review of the issues; for our purposes, see especially its Sect. 5.3.)

In particular, this applies to our topic, causal set theory. For as we shall see in detail (Sects. 5.5 to 5.7): causal set theory seeks to define an appropriate class of causal sets each of which underpins an *essentially unique* solution of general relativity (Lorentzian manifold). Here, common synonyms for 'underpins' are 'recovers' and 'reduces.' So 'reduces' is used, not only for the relation between theories, where each theory has many solutions; but also for the relation between individual solutions or states, from one in the reducing theory, to one in the reduced/emergent theory.[15]

There are two points here, which will be centre-stage from Sect. 5.2 onwards: and which I explain in Sect. 5.2.1. They are both about the fact that the scale that is relevant to a causal set's recovery of a Lorentzian manifold is *before* the continuum limit. I stress that here 'before' has no temporal meaning; it just means 'away from,' i.e., at some non-zero length-scale. But this usage is established; so I will continue to say 'before.'

5.2.1 Before the Limit

The first point is that I italicised 'essentially unique' in order both: (a) to signal that this needs to be made precise (Sect. 5.5 et seq.); and (b) to signal, for philosophers, that uniqueness would be the idea of supervenience, and that essential uniqueness is, rather, the idea of *approximate supervenience*.

That is: just as in philosophy, to say that the $\mathcal{T}$-facts supervene on (are determined by) the $\mathcal{B}$-facts is to say that once the $\mathcal{B}$-facts are set (given), so are the $\mathcal{T}$-facts ('there is no wiggle-room'); so also in causal set theory: to say that a causal set underpins a unique solution of general relativity (Lorentzian manifold) would be to say that once a specific causal set (in the appropriate class) is given, so is a Lorentzian manifold—there is 'no wiggle-room,' for variety about the manifold.

But I stress that causal set theorists do *not* seek such uniqueness, such complete elimination of wiggle-room or variety.[16] For as we shall see, a causal set has structure at the Planck length and at larger scales (in length and time), whereas a Lorentzian manifold has structure on all length(time)-scales, no matter how minuscule. So a causal set can only be expected (no matter how one might define the appropriate class of them) to underpin, or recover, a Lorentzian manifold's structure at the Planck length and above (i.e., at longer lengths and times). It cannot encode structure on smaller scales, and so cannot discriminate between two Lorentzian manifolds that differ in their structures only at some such smaller scale. Thus the aim of capturing manifold-structure at the Planck length and above, but being silent about it below the Planck length, is expressed in the phrase 'essentially unique': which, all hands agree, needs to be made precise—as remarked in (a) above.[17]

The second point develops this idea of approximate supervenience. My mention of the Planck length hints that there is to be a limiting relation between causal sets and Lorentzian manifolds (more precisely: between appropriate classes of them; or if one prefers, between appropriate classes of theories of them); and that the relevant parameter in taking this limit is to be the Planck length and its going to zero (or equivalently: the Planck energy going

to infinity). True enough. But as we will see (Sect. 5.4 onwards), the scientific and philosophical focus is *not* on what happens at the limit—though the summary I first gave (in (1) and (2) at the start of Sect. 5.2) makes one expect it to be there.

Rather, the focus is on what happens *before* the limit.[18] More precisely: what matters, both scientifically and philosophically, is that the novel structures and their associated phenomena (the emergent behaviour) that are rigorously deduced to occur in the limit are foreshadowed in the regime before the limit. That is: a "rough version," or an analogue, of the emergent behaviour is seen *en route* to, i.e., before, the limit. This happens in the examples, often drawn from thermal physics, treated in the references given in Footnote 14.

As I put it in Butterfield (2011b, Sect. 1.2): for a parameter N that tends to infinity (for example the number of atoms in a gas sample), though $N = \infty$ is physically unreal: "There is a weaker, yet still vivid, emergent behaviour that occurs before we get to the limit, i.e., for finite N. And it is this weaker behaviour which is physically real."[19] And the same is true of this Chapter's case-study, causal set theory. For it proposes that a causal set in the appropriate class whose discreteness scale is set at the physically real, i.e., actual, Planck length (viz. Planck length $L_p := \sqrt{\frac{G\hbar}{c^3}} = 1.6 \times 10^{-33}$ cm; equivalently, at Planck energy $E_p = 1.2 \times 10^{19}$ GeV) exhibits a rough version, or analogue, of the continuum geometry of a Lorentzian manifold—where the rough version or analogue is of course understood in terms of restriction to facts about the continuum geometry above the Planck length. (Again, all hands agree that this 'restriction to facts above the Planck length' needs to be made precise; cf. (a) above.)[20]

These two points can be summed up in terms of an image: an image that connects with another familiar idea

(and jargon) in physics, viz. *coarse-graining*. Thus we think of coarse-graining as loss of information by setting aside the values of some quantities. In the paradigm case of classical statistical mechanics, a phase space Γ, whose elements (microstates) are given by the values of very many quantities (treated as real-valued functions on Γ), is partitioned into cells, i.e., equivalence classes (macrostates). A cell or macro-state is given by the values of a small number of quantities: an elite minority which are typically collective quantities, e.g. totals or averages of quantities on the microscopic constituents. So a cell or macrostate is given as the intersection of level-surfaces of this small set of (typically collective) quantities. If we think of Γ as a region of the plane, e.g. a rectangle, then it is partitioned into mutually exclusive and jointly exhaustive sub-regions. And just as fixing a point in the region Γ fixes which sub-region it is an element of, so also fixing a microstate fixes the macrostate: in the metaphor I used above, there is no wiggle-room.[21]

So the above two points, about emergence before the limit, amount to a warning about this image. Namely: it is the image of the scenario, i.e., the relation between the two theories T_t and T_b *in the limit*, where reduction (or supervenience) rigorously obtains. It is *not* an image of the scenario *en route* to the limit. That is: It is not an image of what I called 'approximate supervenience'; i.e., in causal set theory, the scenario of each causal set in the appropriate class determining only an 'essentially unique' Lorentzian manifold—where, again, all hands agree that 'essentially unique' needs to be made precise. In this scenario, to fix a causal set is to fix—not a single Lorentzian manifold—but a set of them that are approximately isometric in the sense of matching, within appropriate error-bars, each other as regards the facts about continuum geometry above the Planck length.

So returning to the phase space Γ and our image of it as a region of the plane: this scenario corresponds, not to a single partition of Γ into sub-regions, but to a set of partitions, so that to fix a causal set is to fix—not a single cell (Lorentzian manifold, macrostate)—but a set of cells. Besides, the boundaries between the cells in one partition of Γ are to be suitably close, in some sense, to the boundaries between suitably corresponding cells in another partition of Γ. And here again, 'suitably close' and 'suitably corresponding' are to be made precise in terms of the cells (the Lorentzian manifolds, the macrostates) approximately matching each other as regards long length-scale facts about continuum geometry.

Finally, it is worth stating this last idea in terms of how (as I mentioned) statistical mechanics defines a cell or macrostate as the intersection of level-surfaces of a small set of (typically collective) quantities. For as we shall see (Sect. 5.7), causal set theory formulates this last idea in a very similar way. An appropriate causal set determines an "approximate isometry class" of Lorentzian manifolds, rather than a unique such manifold; (and so causal set theory works with a set of partitions). That is: causal set theory envisages that an approximate isometry class of Lorentzian manifolds will be given by an approximate intersection (in some sense) of level-surfaces of appropriate functions on the set of causal sets: where for a function to be appropriate, its values must suitably encode some long length-scale facts about continuum geometry.[22]

5.3 Spacetime Functionalism

This Chapter's second background theme, spacetime functionalism, is the application to spacetime of a philosophical 'ism,' 'functionalism,' that arose in the 1960s' debate about the mind-body problem, i.e., the problem of

how mental and bodily facts (facts involving mental and bodily concepts) are related. For example, is the mind, i.e., the realm of mental facts, separate from, though interacting with, the body, the realm of bodily facts: like two nations, say France and Italy, which are neither a part of the other but have many and varied (e.g. causal and legal) interactions? (This is usually called 'dualism.') Or is the mind reducible to the body? That is: is the realm of mental facts really a part of the realm of bodily facts: like a *Département* is a part of France? (This is often called 'materialism.') As my phrasing, 'one realm being a part of another' suggests, materialism is often formulated more precisely in terms of reducing (cf. the start of Sect. 5.2) a theory of mind, a theory of the mental facts, to a theory of bodily facts.

Being an 'ism', the word 'functionalism' was vague, with different meanings for different authors. But broadly speaking, most took it as analogous to emergence in the more general reduction/emergence contrast: mental facts, or a theory of them, "say something additional to" bodily facts, or a theory of them—they "have a life of their own." A bit more precisely: the idea of functionalism was twofold.

(A) One should philosophically understand mental concepts such as pain or belief by focussing on the concept's web of relations to other concepts, both mental and bodily (this web being called the concept's *functional role*).

(B) More specifically: by focussing on functional roles, one could see correctly how mental facts 'fitted in the landscape' of bodily facts: they are not a separate realm *à la* dualism, nor are they merely a sub-realm of the bodily facts, i.e., reducible to them.

But beware: (B) is rough speaking. As we will see in a moment, one very influential formulation from the mid-1960s (viz. Lewis' (1966, 1970, 1972, 1994)), which was

later called ‘functionalist,’ explicitly allowed that, while (A) is indeed true, there is nevertheless a reduction.

Accordingly, spacetime functionalism is the doctrine that one should philosophically understand a spatiotemporal concept, especially a chrono-geometric concept (and even the concept of spacetime itself) by focussing on the concept’s web of relations (again called *functional role*) to other concepts, both spatiotemporal and material (where ‘material’ includes radiation as well as matter). This doctrine was formulated (and named) by Knox (2014, Sect. 2; 2019, Sect. 4). These papers advocated a specific version saying that the functional role of spacetime itself is (no more than) to define a structure of local inertial frames (Knox, 2019, 122): a version which has faced criticisms (e.g. Read & Menon, 2021), But the general idea of spacetime functionalism has garnered a lot of interest, for two reasons.

First, the chrono-geometric/material contrast (echoing the mind/body contrast) obviously relates to the perennial debate between substantivalist and relationist conceptions of space and spacetime. So it has been natural to ask whether relationist doctrines to the effect (roughly speaking) that the physics of matter and radiation determines, and maybe even explains, chrono-geometry should be understood and assessed in terms of functionalism. And similarly, it has been natural to relate Brown (2005)’s dynamical account of chrono-geometry to functionalism.

Second (and more relevant here), it has been natural to ask whether proposals by quantum gravity physicists about ‘the emergence of classical spacetime’ from their posited non-spatiotemporal fundamental degrees of freedom should be understood and assessed in terms of functionalism. Indeed, this Chapter’s topic—causal set theory’s endeavour to recover Lorentzian manifolds using causal sets—has

been a case-study of this, see Lam and Wüthrich (2018, Sect. 4; 2021, Sect. 4), Huggett and Wüthrich (forthcoming, Chaps. 2.4 and 3.3).

So much by way of introducing spacetime functionalism. I myself have two stakes in this game: as explained in each of the next two Subsections.

5.3.1 Lewis' Position About Mind and Body

First, Gomes and Butterfield (2023, Sects. 4 and 5) have urged philosophers to recall what Lewis' position really was (whether or not they agree with it, of course). For the subtlety and precision of his position has been lost in the slew of literature in the philosophy of mind, advocating some non-Lewisian versions of functionalism as the best form of non-reductive materialism. Thus Lewis (1966, 1970, 1972, 1994) filled out (A) and (B) (at the start of Sect. 5.3) as follows. I will present his proposals using the case of mind and body. But it will be clear that they can apply perfectly well to theories about other subject-matters, including spacetime theories; (this generality is very clear in Lewis 1970).

Lewis' first point is that (A) misses a trick. It is not just, as (A) says, that one can specify a mental concept by its functional role; for after all, almost all concepts can be thus specified. Also, one can *simultaneously specify* several concepts by their roles, even though each of these roles mentions some or all of the other concepts. Think of how the role of pain includes not just relations to bodily concepts, like being typically caused by tissue-damage, and causing aversive behaviour, but also relations to other mental concepts (like typically causing distress)—whose roles mention pain. This interconnectedness of functional roles makes one fear a vicious logical circle of definition. But Lewis showed in detail that all can be well: such simultaneous specification can avoid a vicious circle.

Furthermore, Lewis pointed out that (B) is *compatible* with a reduction. For after we assert our theory of mind and body, our *melange* of mental and bodily propositions ('the long conjunction of platitudes of everyday psychology', as Lewis puts it (1972, 256)) from which one extracts functional roles, and which Lewis then shows (as I just reported) to provide simultaneous specifications of all the mental concepts: we may perhaps develop *a second theory that contains the same functional roles*—but which also describes the occupants of those roles (i.e., the referents of the long definite descriptions, 'the *F*', that express the roles) using *other* concepts (in linguistic terms: in other vocabulary), than occurred in our first theory, i.e., in our *melange* or long conjunction.

So in terms of my mnemonics: our first theory is T_t, and the second theory is T_b. And I wrote 'after we assert' only for brevity. For the temporal order is not relevant: what matters is that the second theory is logically independent from the first. Of course, for the case of mind and body, this development of the second theory has indeed happened, thanks to the rise of neurophysiology. For that science has theories in which pain, i.e., the occupant of the pain-role, is —not just uniquely specified simultaneously with other concepts, *à la* Lewis—but also described with neurophysiological (not just bodily) vocabulary. (The philosophical literature, ignoring the contingent scientific details, abbreviates such a description as 'C-fibre firing.')

Lewis shows that in such a situation, there *is* a reduction in the Nagelian sense of a deduction of T_t from the conjunction of: (i) T_b and (ii) the statements connecting each of those concepts in T_t that are specified by their functional role in T_t, to T_b. That is: there is a deduction from the conjunction of T_b with the bridge laws (cf. (1) at the start of Sect. 5.2). But notice that in this setting, the notion of reduction is logically stronger than the original

Nagelian notion, in the sense that this setting requires (while Nagel does not) simultaneous specifications of concepts by their inter-connected functional roles; indeed both at the bottom and at the top level.

Accordingly, Lewis does *not* claim that this logically stronger reduction is bound to be possible: he claims only that in some cases, it is—including, he contends, the case of mind and body. He also discusses cases where nothing exactly fulfils a given functional role, but something nearly does, and does so better than anything else. This sort of case, of *near-realizers*, is of course logically weaker than the "pure" Lewisian proposal which (for clarity) I first expounded. So this sort of impure case is more widespread —and I will return to it in Sect. 5.8.2.

Lewis also emphasises that in such a reduction, the bridge laws are mandatory, not optional. They are contingent statements, of the identity of an object (or of co-extensiveness of properties). But they are conclusions of a deductive argument, whose premises come from both theories, T_t and T_b. They are not hypotheses or verbal stipulations motivated by some methodological virtue such as theoretical simplicity; (as previous advocates of such a reduction, often called 'mind-brain identity theory,' had said).

For later comparison with spacetime cases, it is worth summing up this discussion using the standard mind-body example of a bridge law: 'pain is C-fibre firing.' The point will be that it is derived: not guessed nor stipulated. Thus consider:

(i): Accepting the characterization of pain given by everyday psychology (its being typically caused by tissue-damage etc.), we endorse the premise: pain is the unique occupant of so-and-so role.

(ii): Accepting neurophysiology, we endorse the premise: C-fibre firing is the unique occupant of so-and-so

role. (Here again, 'C-fibre firing' is the ignorant philosopher's catch-all for a technical description using the vocabulary of neurophysiology.)

(iii): So by the transitivity of identity (i.e., 'if $x = y$ and $z = y$ then $x = z$'), we must infer: pain *is* C-fibre firing.

So much by way of reporting Lewis' proposals. For more details (and for the contemporaneous though less precise and detailed positions about mind and body, of Armstrong and Putnam), see Butterfield and Gomes (2023, Sects. 1.1, 2.1, 4 and 5). Gomes and I thus proposed a label for a reduction that ensues when the second theory, T_b, has the same roles (with occupants differently described) as are simultaneously specified in T_t. Namely, we call it a *functionalist reduction*.

5.3.2 Functionalist Reduction for Spacetime Theories

Our point in that paper was not just an admonition about the literature's treatment of the mind-body problem. More important (and this is my second 'stake in the game'): we also argued that spacetime theories give well-worked out examples of functionalist reduction (which we presented in Gomes & Butterfield, 2022, 2024). And so we accused the literature about spacetime functionalism of missing a trick, i.e., of not noticing that Lewis' proposals were a "golden oldie" predecessor. More's the pity, since as I said at the start of Sect. 5.3.1, it was always clear (especially in Lewis, 1970) that the proposals can apply perfectly well to theories about subject-matters other than mind and body.[23]

Thus, for this Chapter, the question arises: is causal set theory *en route* to providing, not just a Nagelian reduction, but a functionalist reduction, of Lorentzian geometry?[24] To prepare for addressing that question (in Sect. 5.8.2), I should here review one such well-worked out example. My example is simultaneity in special relativity. More

specifically: my example is the definition of simultaneity as a relation between spacetime points, in terms of their being causal connectible (i.e., connectible by a signal at most as fast as light). As the mention of causal connectibility suggests, this example will be very relevant in Sect. 5.5.[25] There are two key points, which will yield (respectively) premises analogous to those in (i) and (ii) in Sect. 5.3.1; and so the derivation of a bridge law analogous to 'pain *is* C-fibre firing.'

First: using the ideas of special relativity, one can prove there is a *unique* equivalence relation on the spacetime points satisfying conditions that, most physicists would agree, are part of the meaning of the term 'simultaneity.' The idea here is to go beyond saying 'two points are simultaneous iff they cannot be connected by any signal no matter how fast.' Agreed: in a Newtonian spacetime with arbitrarily fast signals, this suggestion is true; for it specifies simultaneity as the two points being in the same absolute 3-dimensional hyperplane, as one intuitively wants. But in Minkowski spacetime, understood as limiting any signal's speed to that of light, this suggestion specifies simultaneity as identical with being spacelike-related—which is not an equivalence relation.

However, we can go beyond this suggestion, and secure simultaneity being an equivalence relation *relative to an inertial observer*, by considering which spatially distant events such an observer would judge simultaneous with events on her worldline, assuming that she uses the radar method (i.e., organizes a light-signal to bounce off a mirror located at the distant event) and takes the speed of light to be independent of spatial direction. These considerations lead, of course, to the famous, indeed revolutionary, frame-dependence, or relativity, of simultaneity, presented in the opening paragraphs of Einstein's 1905 paper. For us, the important points are that:

(a) nowadays, the theory is so well understood and accepted that most physicists agree these considerations are part of the meaning of the term ‘simultaneity’; (cf. above: being caused by tissue-damage etc. is part of the meaning of the term ‘pain’); and

(b) these considerations can be formulated as conditions that are satisfied only, i.e., uniquely, by the (orthodox, textbook) frame-dependent simultaneity relation.

Summing up, we have our analogue of the premise in (i) of Sect. 5.3.1. Namely, a premise (understood as justified by the meanings of the words) along the lines:

(i’) Simultaneity relative to an inertial worldline L = the unique equivalence relation on spacetime points, Sim_L, such that it holds between two points iff an observer on L, using the radar method with an isotropic speed of light, would judge them simultaneous.[26]

The second key point, yielding a premise analogous to that in (ii) of Sect. 5.3.1, is more technical, i.e., less well-known. (So again, this will be analogous to the premise in (ii) being warranted by technical neurophysiology.) Over the years, logicians have developed axiomatic theories of the kinematical structure of special relativity, i.e., of Minkowski geometry, written in formal logical languages, such as predicate calculus. Among these, the theory relevant here is Robb’s theory (1936[1914]). It has a single binary predicate *After*(x, y) (read as: ‘x lies in or on the future light-cone based at y’), subject to a set of axioms so rich that every model (in the logicians’ sense) of the axioms is isomorphic to Minkowski spacetime. So this is a rigorous axiomatization of geometry based on causal connectibility and without any invocation of coordinate systems: it is a synthetic rather than analytic axiomatization, analogous to what Hilbert had done in 1899 for Euclidean geometry.[27]

It turns out that in this theory, simultaneity with respect to an inertial worldline L is explicitly definable in terms of

After, i.e., with no other non-logical vocabulary or concepts being invoked. In other words, *orthogonality* (in the sense of Minkowski geometry) to L *is* causally definable. It is just that the definition is a lot longer and subtler than the suggestion, 'events are simultaneous iff not causally connectible,' with which our discussion began. (For details of the definition, cf. ibid. and Malament (2009, Sect. 3.4)). This provides our analogue of the premise in (ii) of Sect. 5.3.1:

(ii'): Orthogonality to L (causally defined)= the unique equivalence relation on spacetime points, Sim_L, such that it holds between two points iff an observer on L etc.

And thus we derive the bridge law. We *must* identify simultaneity, as understood in terms of an observer's judgments using the radar method ('simultaneity in T_t'), with orthogonality, as given by the long causal definition ('orthogonality in T_b').

That is (iii'): By the transitivity of identity (i.e., 'if $x = y$ and $z = y$ then $x = z$'), we must infer: simultaneity relative to L *is* orthogonality relative to L.[28]

So much by way of illustrating how spacetime theories can illustrate functionalist reduction. As I said, Sect. 5.8 will address the question whether causal set theory illustrates some version of functionalism, such as functionalist reduction.

5.4 The Endeavour of Recovering a Continuous Space

I turn now to the endeavour of recovering a continuous space from a discrete one; or in other words, showing a continuous space to approximate a discrete one. This Section discusses this endeavour in very general terms; (the next Section turns to causal sets). Indeed, in this Section we need not choose some exact definition of

'discrete' and 'continuous': we can make do by keeping in mind the three paradigm cases of—in one dimension—the integers (denumerable, not dense), the rationals (denumerable, dense), and the reals (non-denumerable, dense but indeed continuous). So I take our endeavour to be: to find non-continuous structures that recover (underpin) our current—and supremely successful—models of space and time as continua. A bit more precisely: we take ourselves to have a theory T_c which posits a continuous space and-or spacetime, with (what we believe to be) good features/results *G*, and bad features/results *B*; and we seek a theory T_d with a discrete space and-or spacetime, that (to as high a precision as we can probe): has *G* or most of *G*, and lacks *B* or most of *B*.

5.4.1 An Ever-Finer Mesh?

Prima facie, it is tempting to think that we are guaranteed to succeed in this endeavour, thanks to the finite precision of any measurement we can ever make. One thinks: "the finite precision of measurements of length and time implies that we can reproduce T_c's results with a discrete space or time whose lattice-spacing is below that precision. And so we can succeed in the endeavour." But of course, this implication is not firm. For from the time of Zeno, there has been controversy about the coherence of discrete models of space and time; (I take the development of the calculus, and in particular of the concept of instantaneous velocity, to have resolved controversy about the coherence of continuous models).

The general issue is that there are various ways in which a discrete structure, no matter how minuscule the scale of discreteness, could imply features (either structural features or specific phenomena) that are detectable at a macroscopic scale and contrary to what a continuous model of space or spacetime implies—and that might be

contradicted by our experience. One example is the obstruction to modelling chiral fermions on a discrete spacetime, summed up in the no-go theorem of Nielsen and Ninomiya (see Wipf, 2013, Chap. 15).[29] But even allowing that we are not guaranteed to succeed, it is natural to try by thinking of an ever-finer mesh or lattice. That is, one envisages a sequence of discrete theories T_{d_i} that tends in a suitable sense to the continuous theory T_c. Or in terms of the spaces themselves, rather than theories of them: one envisages a sequence of ever-finer meshes or lattices which tends to a continuous space. The obvious prototypes here are: (i) decimal expansions of numbers that terminate after their *i*th place, and these expansions' limits, the real numbers; and (ii) a sequence of lattice systems, with the lattice spacing going to zero.

The rest of this Section will develop this strategy of formulating an ever-finer mesh or lattice. I will first explain two ways it could be misleading. This will lead in Sect. 5.4.2 to a trio of *desiderata*, and then to a formulation of what 'shape' the recovery of the continuum from the discrete should take. This strategy could be misleading in two related ways; both of which will apply to the case of causal sets. First: note that there is no *a priori* need for a limiting process. For we seek 'only' a discrete theory that obtains (in my jargon above) the good features *G* and avoids the bad features *B*, to within the required precision, i.e., to as high a precision as we can probe. This is a matter of 'getting things right'—more precisely: getting things as right as the continuous theory T_c gets them—*before* the continuum limit, not at it. In other words: to explain our success in modelling space and time as continua—to recover the continuum as an approximation—it is *not* mandatory to obtain a theorem stating that the continuum, or even just its successful good features *G*, rigorously hold in some limit. (But this is not to say that such theorems, if

available, are not useful or not illuminating. Of course, they can be: and again, causal set theory will give examples.)

The second way that this ever-finer strategy could be misleading is subtler. It will be clearest to introduce it by considering a heuristic that the strategy suggests. (It is a heuristic that causal set theory has adopted; so it is relevant to this Chapter.) Namely, the strategy prompts one to seek the desired discrete space or structure by studying meshes/lattices that are defined by *selection* from the continuum. The idea is to select points from the continuous space or spacetime, endow them with whatever continuum notions make sense once restricted to such a 'bare' set of selected points—and then see how these notions, as defined by restriction, fare as 'ingredients' for underpinning or recovering the original continuous space or spacetime.[30] This heuristic is all very well: and in Sects. 5.6 and 5.7, we will see causal set theory adopt it. But it brings out a special feature of the continuum, or more precisely of the idea of a manifold. Namely: much of its mathematical structure, such as it being a Hausdorff topological space or the integer value of its dimension, is defined at all scales, no matter how small.[31]

And the point is: physical space or spacetime might well not be like that. There are, of course, countless possibilities. So in recent decades, mathematicians have investigated and classified many of these possibilities; for example, the idea of a stratified manifold (roughly speaking, a connected topological space that decomposes into manifolds of different dimension). In this Chapter, I will not need details of these possibilities (Anderson (2014) is a physicist's survey). I only need to register their existence. More precisely: if we think of recovering the continuum (or the continuous theory T_c) from some posited discrete structure (or theory T_d), by successively "zooming out" the scale, then we must allow that at a sequence of

intermediate scales, we might see a sequence of different structures, both metric and topological. For example, as we zoom out, the dimension might change as in a stratified manifold.

5.4.2 The Shape of the Recovery

We can sum up this discussion of the 'ever-finer' strategy, as the following trio of statements, (A)-(C), about what we should seek, and what we should *not* seek, when recovering a continuous space from a discrete space; (similarly of course for spacetime).

(A): Since we think of the discrete space and its associated structures (or discrete theory T_d) as representing the true physics which is to recover the continuous space as effective (in other jargon: to recover the successes of the continuous theory T_c): we should *not* require that there is a limit theorem, i.e., a theorem that the continuous space (or T_c, or its good features G) rigorously exist in the limit of vanishing discreteness scale.

(B): On the other hand: we *do* need to demand that the discrete space (or theory T_d) gives no, or at most small, variation in the values of those quantities that are to recover the empirical success of the continuous space (or theory T_c). Typically, this success is on scales that are very large compared to the discreteness scale. And for these large scales, the discrete space's (or theory's) values of these quantities must not differ much from what the continuous space (or theory) describes.[32]

(C): We need to *allow for* (though not of course demand) the discrete space (or theory T_d) having arbitrarily large variation in the values of these quantities, in regimes "far from" the empirical success of the continuous space (or theory T_c): i.e., typically, on very small length scales, close to the discreteness scale. In these regimes and scales, the discrete space (or theory's) values can vary wildly from

what the continuous space (or theory) describes. And similarly, we should allow for wild mathematical structures; see Footnote 32.

The natural way to implement this trio of statements, (A)-(C)—which is, indeed, the way adopted by causal set theory—is to impose them as conditions on an appropriate injective map from the discrete space to the continuous space.

Thus we envisage a continuous space $(M, \mathcal{S}_M)$ (where M a manifold, and $\mathcal{S}_M$ are various structures on M) and a discrete space $(X, \mathcal{S}_X)$ (where $\mathcal{S}_X$ are various structures on X). Then it is natural to say that for $(M, \mathcal{S}_M)$ to be *an approximation to* $(X, \mathcal{S}_X)$ requires that:

(1): there is an injective function $i : X \to M$ that induces, in the usual "push-forward" way, a definition of structures $i^*(\mathcal{S}_X)$ on M;

(2): some sort of coarse-graining procedure *Co* is defined on $i^*(\mathcal{S}_X)$; so that the result $Co[i^*(\mathcal{S}_X)]$ matches, up to the required precision, the structures $\mathcal{S}_M$ that are given on M. We write $Co[i^*(\mathcal{S}_X)] \approx \mathcal{S}_M$; while of course accepting (cf. the previous Subsection) that the notation $\approx$ needs to be made precise, e.g. in terms of small-enough differences in the values of quantities.

In effect, the formulation (1) and (2) encodes the idea that a recovery, or underpinning, of $(M, \mathcal{S}_M)$ *exists*, viz. by $(X, \mathcal{S}_X)$. Besides, this formulation does not require a limit theorem: which accords with (A) above.

But there are questions—to put it in mathematical jargon—of uniqueness, as well as of existence. That is to say: we also want $(X, \mathcal{S}_X)$ to be 'rich' or 'logically strong' enough to fix an 'essentially unique' $(M, \mathcal{S}_M)$. Recall again Footnote 32, and the discussion of *approximate supervenience* in Sect. 5.2.1. That is: it is natural to require the *approximate uniqueness*, of $(M, \mathcal{S}_M)$, in the following sense; which I label '(b)' so as to match the label (B) above:

(b): If $i_1 : X \to M_1$ and $i_2 : X \to M_2$ are two embeddings of $(X, \mathcal{S}_X)$ satisfying (1) and (2), then the image spaces $(M_1, \mathcal{S}_{M_1})$, $(M_2, \mathcal{S}_{M_2})$ are related by an approximate isomorphism, say $\theta : M_1 \to M_2$, of the structures $\mathcal{S}_M$, where θ respects the embeddings of X in the sense that $\theta \circ i_1 = i_2$. Here again, we of course accept that 'approximate isomorphism' needs to be made precise.

On the other hand, we should *not* want the corresponding approximate uniqueness of $(X, \mathcal{S}_X)$, for given $(M, \mathcal{S}_M)$. That is: we do not want the following condition; which I label '(c)' so as to match the label (C) above.

(c): if $i_1 : X_1 \to M$ and $i_2 : X_2 \to M$ are embeddings of $(X_i, \mathcal{S}_{X_i})$ satisfying (1) and (2), then the domain spaces $(X_1, \mathcal{S}_{X_1})$, $(X_2, \mathcal{S}_{X_2})$ are related by an approximate isomorphism, say $\theta : X_1 \to X_2$, of the structures $\mathcal{S}_X$, where θ respects the embeddings of X in the sense that $i_2 \circ \theta = i_1$. In short: condition (c) is *not* justified, since we should allow the discrete space to be "wild," and so various, on scales so microscopic as to be beyond the regime of $(M, \mathcal{S}_M)$'s empirical success.

5.5 Causal Sets: The *Hauptvermutung*

So much by way of discussing the general endeavour of recovering a continuous space from a discrete one. I turn to how causal set theory implements the ideas of Sect. 5.4. Recall from Sect. 5.1 that I specialise to Lorentzian manifolds: I set aside causal set theory's treatments of matter and radiation, of quantum theory, and even of dynamics. This will mean that Sect. 5.4.2's idea of approximate isomorphism will boil down to approximate isometry for Lorentzian manifolds.

So for my limited purposes,[33] the following five points will suffice as an introduction to causal sets.

(1): A *causal set* is a partially ordered set (poset) $(C, \prec)$ that is locally finite in that for any *x*, *y*, the order-interval (Alexandrov set) $\{z : x \prec z \prec y\}$ is finite.

(2): The elements of *C* are interpreted as point-like events ("atoms") in a discrete spacetime, and $\prec$ represents causal connectibility.

(3): So the proposal is that this is an adequate basis for recovery—in philosophical terms, for reduction—of Lorentzian geometry, with its manifold structure, metric and derivative notions like geodesics and curvature. This is bound to seem a tall order, even to someone sympathetic to the idea of a causal theory of time and aware of Robb's causal axiomatisation of special relativity (cf. Sect. 5.3.2, especially Footnote 27). After all, familiar rigorous accounts of the empirical basis of Lorentzian geometry (such as Ehlers, Pirani and Schild (1972), Adlam et al. (2022), Linnemann and Read (2022)) invoke a much richer set of primitives than just a binary relation of causal connectibility.

(4): But there is a theorem supporting the proposal, due to Hawking et al. (1976), and Malament (1977a). It says that in general relativity, causal structure *does* indeed determine almost all the manifold *and* metric structure. Here, we must restrict 'general relativity' to Lorentzian manifolds that are of dimension ≥ 3, and *distinguishing*—which is a mild restriction of good causal behaviour, viz. that at every point $p \in M$, every neighbourhood of *p* has a sub-neighbourhood which no future-directed, nor any past-directed, non-spacelike curve through *p* intersects more than once. And my phrase 'almost all' signals that the metric is fixed up to a conformal factor i.e., a dilation of the metric varying from point to point (so given by a positive scalar on the manifold). Thus the theorem is: Given two distinguishing Lorentzian manifolds, $(M, g), (M', g')$: if a bijection $\psi : M \to M'$ is a causal isomorphism, then M, M'

are diffeomorphic and ψ is a smooth conformal isometry, i.e., an isometry upto a conformal factor.

(5): Then causal set theory proposes to recover the conformal factor in the metric, by an idea reminiscent of Riemann's suggestion in his great *Habilitationschrift* (1854) that in a discrete space, counting gives a uniquely natural measure. That is, one recovers the conformal factor (giving the volume of spacetime regions) by having the causal set's points be embedded in the Lorentzian manifold at a density of one per Planck spacetime-volume. In "lab units," this is of course a very high density, viz. about 7×10^{138} cm^{-3}. sec^{-1}. (More precisely, in light of the probabilistic ideas developed in Sect. 5.6 below: this is to be the *mean* density.) This proposal is in causal set theory's founding paper: 'When we measure the volume of a region of spacetime, we are merely indirectly counting the number of 'point-events' it contains. No attempt to "pack more points into the same volume" could change their density, because it would only increase the physical volume of the region in which they were placed' (Bombelli et al., 1987, 522). We can sum up (1) to (5), especially (4) and (5), in the slogan: *Causal order + Number = Geometry*.

5.5.1 The *Hauptvermutung for Lorentzian Manifolds*

With these five points in hand, we can state what has come to be called the *Hauptvermutung* ('main conjecture') of causal set theory: which is, in effect, the theory's version of Sect. 5.4.2's conditions (1), (2) and (b) for the recovery of a continuous space from a discrete one.[34] This *Hauptvermutung* was formulated already in the founding paper by Bombelli et al. (1987). First, they give a definition; (albeit not a mathematically exact one, since the probabilistic idea of a *mean* unit density per Planck spacetime-volume was only sketched, viz. in their Footnote

9—but the idea is developed in Sect. 5.6 below). They say that an injective function $i : CM$ from a causal set $(C, \prec)$ to a Lorentzian manifold (M, g) is a *faithful embedding* iff:

1] it preserves causal relations; i.e., writing as usual the causal past of a point $p \in M$ as $J^-(p)$: $i(x) \in J^-(i(y))$ iff $x \prec y$;

2] $i(C)$ is distributed uniformly, with unit density per Planck spacetime-volume, in (M, g): (but allowing for Poisson-type fluctuations);

3] the characteristic length λ over which the geometry of (M, g) varies appreciably is everywhere much greater than the mean spacing between embedded points.

Evidently, this definition's clause (1) implements Sect. 5.4.2's condition (1), about "pushing forward" discrete structures so as to recover a continuous space. As to Sect. 5.4.2's condition (2), about a coarse-graining procedure *Co*, Bombelli et al. (1987) also have this idea.[35] It is not just that their clause (2) avoids the mistake of lacunae, i.e., of omitting important features of the macroscopic structure; cf. Footnote 30. Also their clause (3) obviously corresponds to Sect. 5.4.2's statement (C). That is: they explicitly allow the discrete space to have a wild or various microscopic structure, undetected in our macroscopic measurements—by combining clause (2)'s mean unit density in natural units, with clause (3). Thus their Footnote 10 (p. 524) says (in my notation):

> Notice that clause 2] on the density would not help us to detect a unique approximate metric if we did not also have clause 3] on the characteristic length λ : Given any manifold with the right causal structure, i.e., conformal metric, we could always arrange the density to be unity by setting the conformal factor appropriately; but in doing so we would in general introduce an unreasonably large curvature, or other

> small characteristic lengths. However, it seems plausible that clauses 2] and 3] alone determine the continuum geometry "up to arbitrary variations on small scales, and small variations on arbitrary scales" (where small scale means size unity or smaller).

The last sentence of this quote is in effect a summary of the *Hauptvermutung*; which is formulated in their main text as follows (in my notation, Bombelli et al. 1987, 522):

> ... [if a faithful embedding of a given causal set $(C, \prec)$ into some Lorentzian manifold (M, g) exists], then our discussion up to now leads us to expect that it is essentially unique. In other words, we can expect that any pair of faithful embeddings $i_1 : C \to (M_1, g_1)$ and $i_2 : C \to (M_2, g_2)$ are related by a C-preserving diffeomorphism $\theta : M_1 \to M_2$, which is an approximate isometry of g_1 to g_2. (By C-preserving, we mean $i_2 = \theta \circ i_1$). A precise formulation and proof of this statement would establish rigorously that the continuum approximation is well defined, and therefore that a causal set has a structure rich enough to imply all the geometrical properties we attribute to continuous spacetimes.

Obviously, this formulation echoes Sect. 5.4.2's statement (B), and its corresponding condition of approximate uniqueness, (b). Recall again Footnote 32, and the discussion of approximate supervenience in Sect. 5.2.1. Accordingly, since this formulation in 1987, the hunt has been on to find a precise definition of *approximate isometry* between Lorentzian manifolds, that provides a precise formulation and proof of the *Hauptvermutung*. So I now briefly review this hunt: looking first at a geometric strategy (Sect. 5.5.2), and then at a probabilistic strategy (Sects. 5.6 and 5.7). As I announced at the end of Sect. 5.1,

there are several “success stories,” especially for the probabilistic strategy.[36]

5.5.2 A Lorentzian Manifold as the Completion of a Set of Causal Sets

Bombelli and Meyer (1989) pursue a version of Sect. 5.4.1’s strategy of an ever-finer mesh, using heuristics drawn from classical analysis’ completion of the rationals to give the reals. They succeed in the sense that they prove that a given Lorentzian manifold is completely recovered from the causal sets that it defines. So this is our first success story; (though their description is a sketch, citing unpublished work). They proceed in two stages. The first stage combines the idea of a *direct limit* with the Hawking-Malament theorem (recall (4) in Sect. 5.5). Recall that if:

(i) $(I, \leq)$ is a directed set (i.e., $\leq$ is a reflexive transitive relation on I such that any two elements have an upper bound), and

(ii) $\{A_i : I \in I\}$ is a family of algebraic objects (e.g. groups, vector spaces, or as in our case: posets) indexed by I, with homomorphisms $f_{ij} : A_i \to A_j$ for all $i \leq j$, that obey: $f_{ii} = id_{A_i}$ and $f_{ik} = f_{jk} \circ f_{ij}$ for $i \leq j \leq k$:

Then, the *direct limit* of the family $\{A_i\}$ is defined to have as its underlying set the disjoint union of the A_i, modulo the equivalence relation, written $\sim$, defined by: $x_i \sim x_j$ iff there is a k with $i \leq k$, $j \leq k$ such that $f_{ik}(x_i) = f_{jk}(x_j)$.

So the intuitive idea is that two elements in the disjoint union are equivalent if and only if they “eventually become equal.” So given a Lorentzian manifold (M, g), Bombelli and Meyer consider the set, $\mathcal{C}(M)$, of all the causal sets embeddable in M. These causal sets are then the algebraic objects, ordered by subposethood (i.e., with the f_{ij} being injective $\prec$-preserving functions), that obey (ii) above. And

although in the disjoint union, a point in the manifold gets dis-identified into many copies, across the many causal sets (considered as subsets of the manifold) that it is an element of, the quotienting by the equivalence relation $\sim$ re-identifies these copies. Thus the direct limit of the family $\mathcal{C}(M)$ *is* the points of M, endowed with their causal relations.

Now Bombelli and Meyer invoke the Hawking-Malament theorem. That is: Since $\mathcal{C}(M)$ determines the points of M endowed with their causal relations, it also determines the topological and differential structure of the manifold, and also the metric except for a local conformal (or volume) factor.

The second stage addresses recovering the conformal factor. It consists of two steps. First, motivated by the idea that spacetime volume is given by the number of causal set points (recall (5) in Sect. 5.5) they consider the subset of $\mathcal{C}(M)$ consisting of those causal sets that are uniformly embeddable in (M, g). They then consider a sequence of such causal sets, each of which is a subposet of the next and each of which is uniformly embeddable in (M, g) (with increasing densities, of course). They also here invoke probabilistic ideas: they envisage each causal set in the sequence being the outcome of a Poisson process (which means, roughly speaking: independent trials with equal spacetime volumes getting equal probabilities). This implies that the direct limit of such a sequence is a causal set, written C_ω, such that any given point in the original manifold M has only zero probability of being in C_ω. This last fact means that in order to get M, C_ω must somehow be completed.

So in the second step, Bombelli and Meyer show that for an arbitrary poset P, there is a Dedekind-cut-like construction of the completion of P, $\bar{P}$, that uses the sets of upper bounds of increasing sequences of points of P, and

the sets of lower bounds of those sets of upper bounds. By applying this construction to C_ω, they then show that with probability one (again, in the Poisson-process sense), the completion $\bar{C}_\omega$ consists of the points of M endowed with their causal relations.

So as before, by the Hawking-Malament theorem: the topological, differential and conformal structures are recovered. But now (unlike the end of the first stage): since each causal set in the chosen sequence is uniformly embedded, there is probability one that the conformal factor at a dense set of points is determined. So by continuity, it is determined everywhere. Thus *all* the information in the Lorentzian manifold (M, g) is recovered.

But we should note that this success in recovering the Lorentzian manifold is limited in two ways. First, the discussion is entirely about a fixed, though arbitrary, distinguishing Lorentzian manifold. So it does not give us information about *which* causal sets, among the vast plethora of causal sets, are embeddable in such a manifold. Second: by proceeding to the continuum limit (in each of its stages), it does not relate to Sect. 5.2.1's theme that the regime of primary physical significance is the regime *before* the limit, not at it; specifically, the regime of the Planck scale and above.[37] Sects. 5.6 and 5.7 will report some efforts to address these limitations.

5.6 Poisson Sprinkling: A Statistical Distance Between Geometries

So far, I have said about *Poisson sprinkling* only that (i) it is a process of selecting points of a given (M, g) at some density, with independent trials, and equiprobability for equal spacetime volumes (hence 'Poisson') and (ii) endowing the selected points with their spacetime causal order—thus producing a causal set that is random in the

sense of the Poisson process. (See Sect. 5.1 and the mention in Sect. 5.5.2.) The aim of this Section is to present the proposal (Bombelli, 2000) to use Poisson sprinkling to define a *distance function* on (pairs of) Lorentzian geometries. Section 5.6.1 presents the proposal; then Sect. 5.6.2 reports how it provides suggestive formulations of the *Hauptvermutung*.

5.6.1 Bombelli's Proposal

To present this proposal, I will not need precise definitions of the notions of independence and equiprobability (for which see Bombelli, 2000, Sect. 2). But I should note two general points. First: by Poisson sprinkling into manifolds (M, g) whose geometry varies only over length scales much greater than the Planck length, we get a rich source of faithful embeddings, in the sense of Bombelli et al. (1987); (and defined in Sect. 5.5.1). Thus Poisson sprinkling has been an important tool for investigating which kind of causal set, in the vast multitude of them, can recover Lorentzian manifolds; and so also, which sub-kinds recover specified kinds of Lorentzian manifold; and so also, for investigating the *Hauptvermutung*.

Second, this raises the question whether Poisson sprinkling "covers" the faithfully embeddable causal sets. That is: does a causal set faithfully embed in (M, g) *only if* it is a typical outcome of a Poisson sprinkling? This is not known, but there is positive evidence for it. Thus Saravani and Aslanbeigi (2014) show that, roughly speaking, causal sets that are typical outcomes of a Poisson sprinkling are best at realising the number-volume correspondence, i.e., clause 2] of the definition of faithful embedding. Indeed, they do so even for arbitrarily small regions; (2014, Theorem 1).

I said above that Bombelli defines a distance function on (pairs of) Lorentzian *geometries*. Here 'geometries' means

isometry classes of Lorentzian manifolds (see Footnote 2 above). For the function will have the merit of taking the same value whichever Lorentzian manifolds we choose as representative elements of the isometry classes. The function will also have the merit of being *scale-dependent* in such a way that it expresses insensitivity to structure below the given scale (for us, the Planck scale): just as we would want for the *Hauptvermutung*.

Bombelli's proposal has two main ideas (Bombelli, 2000). First, he associates to each appropriate Lorentzian manifold (M, g) (equivalently: Lorentzian geometry) the probability distribution over finite causal sets given by the Poisson sprinkling process into that manifold. More specifically, he works with distinguishing Lorentzian manifolds (M, g) of finite volume V_M; and for each positive integer n, he considers the set $\mathcal{C}_n$ of causal sets (with the points unlabelled) C, each with n elements. Using the definition of the Poisson process, he deduces a formula for the probability, written $P_n(C/M, g)$, that a Poisson sprinkling from (M, g) yields the unlabelled causal set C (his Eq. 8). Due to the finite volume, the integer n behaves like a discreteness parameter, with larger n implying a smaller discreteness scale ("a finer grain") V_M/n. So the upshot is that we are to think of the manifold (M, g) in terms of the sets of probabilities $\{P_n(C/M, g)|C \in \mathcal{C}_n\}$ that it defines; and as n increases (so that V_M/n decreases), the set of probabilities $\{P_n(C/M, g)|C \in \mathcal{C}_n\}$ gives increasing information about (M, g).

So the task now is to define a notion of distance between two such sets of probabilities (with a common n) for two manifolds (M, g) and (M', g'), i.e., a distance between $\{P_n(C/M, g)|C \in \mathcal{C}_n\}$ and $\{P_n(C/M', g')|C \in \mathcal{C}_n\}$: a distance such that whenever it is small, the manifolds are close at volume scales that are large compared to V_M/n and $V_{M'}/n$.

And here is Bombelli's second main idea. He applies a definition, which is standard in probability theory and statistics, of the distance between probability distributions (called the 'Bhattacharyya angle,' or 'statistical angle'). This definition also has the attraction that it has a simple geometric interpretation (which will justify the word 'angle'); as follows.

Think of any total ordering of the elements of $\mathcal{C}_n$; (no matter which one—it can be regardless of the structures of the causal sets in $\mathcal{C}_n$). Given any such ordering, the set of probabilities $\{P_n(C/M,g)|C \in \mathcal{C}_n\}$ defines a vector in the high-dimensional real vector space $\mathbb{R}^{|\mathcal{C}_n|}$: here, I write $|X|$ for the number of elements in a set X. So if we take the square-roots of the components of this vector, i.e., the $\sqrt{(P_n(C/M,g))}$ as C runs through $\mathcal{C}_n$, we get a unit-length vector in $\mathbb{R}^{|\mathcal{C}_n|}$ (Pythagoras' theorem). That is: the $\sqrt{(P_n(C/M,g))}$ are the components of a unit-length vector in $\mathbb{R}^{|\mathcal{C}_n|}$. Then the *statistical angle* between two such vectors (probability distributions), written $d_n((M,g),(M',g'))$, is defined by:

$$d_n((M,g),(M',g')) \;:=\; \frac{2}{\pi}\,\arccos\left[\Sigma_{C\in\mathcal{C}_n}\,\sqrt{(P_n(C/M,g))}.\sqrt{(P_n(C/M',g'))}\right]. \quad (5.1)$$

Thus $d_n((M,g),(M',g'))$ is the angle between two such unit vectors; except that we have rescaled by the pre-factor $\frac{2}{\pi}$ so as to be at most 1. Example: For finite n, the condition $d_n((M,g),(M',g')) = 1$ requires that the argument of the arccos function in Eq. 5.1 is zero, i.e., that there is no $C \in \mathcal{C}_n$ for which both $P_n(C/M,g)$ and $P_n(C/M',g')$ are non-vanishing, i.e., which can be embedded in both geometries.

So to sum up these two ideas: two Lorentzian manifolds are taken to be close iff when we sample the same number of points at random with uniform density in each of the manifolds, the probability of obtaining any given induced causal set is about the same in the two cases.[38]

Before I turn to how this proposal prompts interesting probabilistic formulations of the *Hauptvermutung* (Sect. 5.6.2), let me make two remarks which relate the proposal back to Sect. 5.5.2's result about completing an appropriate set of causal sets so as to recover the entire Lorentzian manifold (M, g). So these remarks involve considering limits as the number n of points goes to infinity. (Both remarks follow Bombelli: after his equation (10), and his Section V, respectively.)

Firstly: since the number $|\mathcal{C}_n|$ of causal sets that can be made out of n points is finite (though it grows very fast—faster than exponentially (Kleitman & Rothschild, 1975), the value of $d_n((M,g),(M',g'))$, for any n, depends on a finite number of parameters, so that it cannot capture all the information in the manifolds (geometries) (M, g) and (M',g'). This implies that $d_n((M,g),(M',g'))$ is not positive-definite, in the sense that there are non-isometric pairs, (M, g) and (M',g'), such that for all n, $d_n((M,g),(M',g')) = 0$. So d_n cannot be a distance function on the infinite-dimensional space of Lorentzian manifolds (nor the space of Lorentzian geometries), in the sense of metric space theory, viz. that $d_n(x,y) = 0$ implies that $x = y$. Thus Bombelli calls d_n a 'pseudo-distance function.'

This prompts one to consider the limit $n \to \infty$, returning one to Sect. 5.5.2's result. That result means that a sequence of causal sets $\{C_n\}$, where each C_n (i) has n elements (i.e., $C_n \in \mathcal{C}_n$) and (ii) is a sub-causal set of the next one, $C_n \subset C_{n+1}$, can be embedded in at most one manifold (M, g) (upto isometry: i.e., in at most one Lorentzian geometry). This means that for any such sequence $\{C_n\}$

$$\lim_{n\to\infty} P_n(C/M,g)P_n(C/M',g') = 0. \tag{5.2}$$

Then although the number of terms in the sum in Eq. 5.1 grows faster than exponentially, nevertheless the limiting

value of $d_n((M,g),(M',g'))$, for $(M,g),(M',g')$ not isometric, could yet be 1. For although many causal sets are embeddable in both (M, g) and (M',g') (see the example after Eq. 5.1), all the products of probabilties in Eq. 5.2 go to zero fast enough to overcome the super-exponentially increasing number of terms in the sum.

In such a situation, $d_\infty((M,g),(M',g')) := \lim_{n\to\infty} d_n((M,g),(M',g'))$ is a distance function, in that it discriminates (does not vanish on) non-isometric (M, g), (M',g'). However, it does not distinguish (like the d_n do not) between different values of the total manifold volume V_M and $V_{M'}$—simply because the number of sprinkled points was a "control parameter" chosen by us as the same for both manifolds. This limitation prompts the second remark.

Secondly, Bombelli shows that by:

(i) making the number n of sprinkled points be itself a random variable, subject to a Poisson distribution in the elementary sense, i.e., for the manifold (M, g), $P_\mu(n) := (e^{-\mu}\mu^n)/n!$, and choosing the parameter i.e., mean μ to be proportional to the volume V_M; and similarly for (M',g') (which can even have a different dimension than M), requiring $P_{\mu'}(n) := (e^{-\mu'}\mu'^n)/n!$;

(ii) relating the parameters μ, μ' to a common length scale l, which is to be a mean spacing between sprinkled points, the same in both manifolds; and

(iii) revising Eq. 5.1 accordingly, by having the argument of the arccos function be also a sum from $n = 0$ to ∞;

We obtain a distance function d_l (now parametrized by the mean point spacing l; cf. his Eq. 41) that compares manifolds of different volumes, and indeed of different dimensions, with the intuitively correct scaling properties.

More specifically, and most relevantly for our purposes: the merit of this distance function d_l is that it has most of

the contribution to the distance between two manifolds come from probabilities about features "around" the mean point spacing *l*. Or in more physical terms, choosing *l* to be the Planck length: the Poisson processes, and so the distance function d_l, give for each manifold (a) greatest probabilistic weight to those (embeddable) causal sets whose cardinality lies in the range $N \pm \sqrt{N}$, where *N* is volume of the manifold in Planck units; and (b) very small weight to causal sets that are much larger than this. In this way, the distance function is constructed to be insensitive to structure on scales below *l*.

Furthermore, returning to Sect. 5.5.2's result: Bombelli also shows that this result implies that if $d_l((M, g), (M', g')) = 0$, then—with probability one (according to the Poisson process)—(*M*, *g*) and (M', g') are isometric. So this distance function is, with probability one, positive-definite.

5.6.2 Probabilistic Formulations of the *Hauptvermutung*

Finally, I report how Bombelli's function d_n prompts interesting probabilistic formulations of the *Hauptvermutung*; (so I now set aside d_l). He suggests three such: of which I will report two (viz. his (ii) and (iii) above his Eq. 12), since they involve notions I have introduced.

Firstly: recall the start of my first remark above, viz. that since $d_n((M, g), (M', g'))$ depends on a finite number of parameters, it is not positive-definite. This prompts Bombelli to conjecture: For any subset of geometries labeled by a finite number of parameters (analogous to the "minisuperspaces" used for spatial geometries), there is a finite *n* such that d_n is a true distance function on this set.

Secondly: Bombelli conjectures that for any two arbitrary distinguishing, finite-volume non-isometric manifolds, (*M*, *g*) and (M', g'), there is a finite *n* such that

$d_n((M,g),(M',g')) > 0$, with $d_n((M,g),(M',g')) \to 1$ as $n \to \infty$[39].

Bombelli calls both these conjectures 'reasonable': with which I think all would concur (though of course the first needs a precise definition of 'geometry labeled by a finite number of parameters'). But they remain unproven—and so, an invitation to mathematicians. Indeed, more generally: I hope this Section and Sect. 5.5 has conveyed how alluring the *Hauptvermutung* is. It expresses an idea that is both natural in itself (cf. Section 5.4.1), and conceptually central to the causal set programme: that causal sets are an adequate basis for recovering—in philosophical terms: reducing—Lorentzian geometry above the Planck scale (Sect. 5.2.1 about the importance of *before the limit*; and Sect. 5.5.1). And yet for thirty-five years, no precise formulation has been settled on as best; let alone proved (Sect. 5.5.2 and this Section).

But this predicament is no reason for scepticism about causal set theory: and not just because a task's being hard should make it *more* alluring for us all. There are also two other reasons. (1): As I stressed in Footnote 34, causal set theory does *not* have to establish some purely geometric formulation of the *Hauptvermutung*. For its recovery of Lorentzian geometry could ensue as a consequence of its proposals about matter, radiation and quantum theory, proposals that this Chapter has set aside. (2): In any case, there is plenty of evidence that some purely geometric formulation along the lines this Chapter considers is indeed true. Much of this evidence uses Poisson sprinkling to turn functions on causal sets into random variables, which can then be assessed as estimators (in the sense of probability theory) of continuum quantities such as the dimension of a manifold—as we will see in Sect. 5.7.

5.7 Estimating Continuum Geometry from a Causal Set

The idea of this Section is that since in a probabilistic framework, any function on causal sets becomes a random variable, a function that we judge to be a discrete analogue of a continuum quantity of interest can be assessed as an *estimator*, in the sense of probability theory, of that continuum quantity. For example, one takes a function that encodes (at least when evaluated on appropriate causal sets that are large enough) a discrete or coarse-grained analogue of a feature of continuum geometry, such as manifold dimension or the length of a timelike curve: a feature we are concerned to recover, albeit only at and above a certain length-scale. For some continuum features, such as causal pasts and futures, or the volume of a spacetime region, the definition of the function is obvious. For example, the volume of a region is to be given by the number of causal set elements it comprises. But in general, it will take ingenuity to formulate the pattern in causal sets that is relevant to the continuum feature: e.g. the pattern of causal connectibility that underpins, i.e., is a harbinger of, the causal set being embeddable in a manifold of a prescribed dimension.[40]

Given such a function, one then assesses it by looking at the behaviour of the function's mean and variance on faithfully embedded causal sets that are Poisson-sprinkled into a given (M, g), as the density ρ of the sprinkling (and so the size n of the causal set) grows. Thus one hopes to show that such a function's statistical properties make it a good estimator of e.g. manifold dimension, or timelike length.

For example, one hopes to show that: (a) its expectation tends to the value of the continuum quantity (in the Lorentzian manifold that one aims to recover); and (b) its

variance decreases appropriately (e.g. tends to zero), for larger and larger causal sets. And again, proving a limit theorem is illuminating for, but not a *sine qua non* of success: for we seek recovery at and above a certain length-scale.[41]

So, writing the estimator function on causal sets as $\mathbf{G}$ ('G' for 'geometry'), and writing $\mathcal{G}$ for the analogous continuum quantity that we take $\mathbf{G}$ to estimate: we envisage the following three steps:

(i): The expected value, $< \mathbf{G} >$ of the random variable, and its limit as $\rho \to \infty$ are calculated; and shown to equal the value of $\mathcal{G}$ in a Lorentzian manifold (M, g).

(ii): The value of $< \mathbf{G} >$ at finite ρ is shown to be close to its limiting value when the discreteness scale set by ρ is small compared to any curvature scale of (M, g). (Again: clause 3] in Sect. 5.5.1's defnition of faithful embedding; and the theme of *before the limit*.)

(iii): The fluctuations around the expected value are shown to be small so that the continuum value of $\mathcal{G}$ (as ascertained within error-bars) can be read off from $\mathbf{G}$ evaluated on a *single* faithfully embedded Poisson-sprinkled causal set C.

Causal set theorists have several such success stories; (though in some examples, the fluctuation analysis (iii) remains to be done, and-or the results are for sprinkling into a flat spacetime). These examples go well beyond the obvious cases such as causal pasts and futures, or the volume of a spacetime region, that I mentioned above.

For brevity, I will report just two, about manifold dimension and the length of a timelike geodesic. These are natural choices since (i) they show ingenuity in the definition of the estimator, (ii) they are readily summarised (and discussed in some of the reviews), and (iii) they have the historical interest of being treated, albeit briefly, in Myrheim's precursor paper (1978). But these are just two

examples: several others are summarised in Sect. 3.4.2 of Butterfield and Dowker (2023).

So first, *spacetime dimension*: several estimators of dimension, based on different underlying ideas, have been studied. The simplest one (in a class that gives, in general, non-integer values and was studied by Myrheim and Meyer) is the *ordering fraction* of the causal set. It is the fraction of pairs of elements that are related. For a causal set Poisson-sprinkled into an Alexandrov interval (i.e., the intersection of a point's future lightcone, with the past lightcone of a point in that cone) of Minkowski spacetime, its expectation is a known monotonically decreasing function of dimension. Thus this function being decreasing reflects the idea that a higher dimension gives more room for points to be unrelated. Furthermore, this function's variance accurately determines the dimension d for large enough causal sets. For details, see Myrheim (1978, 11), Sorkin (1991, 10, 1991a, 11–15, 2005, 11), and Surya (2019, Sect. 4.1).

For *the length of a timelike geodesic*, the obvious suggestion for an estimator is to count the number of elements in the longest chain (i.e., the linearly ordered set of maximum cardinality) between the two corresponding elements in the sprinkled causal set. For the longest chain is the natural analogue of the timelike geodesic. And indeed: for sprinklings in flat spacetime of dimension d, the cardinality of the longest chain multiplied by the fundamental length, L—where $\rho = L^{-d}$ with ρ the sprinkling density—is a good estimator, for long enough geodesics, of the associated timelike geodesic's continuum-length (times a dimension-dependent constant). For details, see Myrheim (1978, 5–8; 11), Sorkin (1991, 11, 1991a, 17; 2005, 12), Brightwell and Luczak (2015, 5–8), Surya (2019, Sect. 4.3). Besides, this good behaviour is reproduced for

curved spacetimes, at least in the continuum limit (Bachmat, 2008, Theorems 1.1 and 1.2).

5.8 Hopes for Reduction

In this concluding Section, I return to philosophy. That is: I will discuss, in Sect. 5.8.1, how the details of causal set theory (Sects. 5.4 et seq., especially Sects. 5.5 to 5.7) illustrate Sect. 5.2's topic of reduction, especially reduction *before the limit*. Then in Sect. 5.8.2, I discuss the prospects for their illustrating Sect. 5.3's idea of *functionalist reduction*.[42]

Broadly speaking, my conclusion will be that the pressure is on the physics, not on the philosophy. That is: causal set theory still has much to do, in its endeavour to recover Lorentzian geometry; (more precisely, the theory of distinguishing Lorentzian manifolds). But its tasks lie in theoretical and mathematical physics, not in philosophy. For nothing in that endeavour undermines, or militates against, the traditional Nagelian account of reduction. Nor does anything in that endeavour undermine or militate against a functionalist reduction: though achieving a functionalist reduction will surely be a harder task simply because, as I said in Sect. 5.3.1, it is a logically stronger concept of reduction. So in short: causal set theory's endeavour to recover Lorentzian geometry has the wealth of ideas and results, the success stories, that I have reviewed in Sects. 5.5 to 5.7. It is indeed *en route* to reduction, but it still has a way to go.

5.8.1 The Prospects for Nagelian Reduction

Recall from (2) at the start of Sect. 5.2 that the Nagelian account of reduction allows the theory that is reduced, i.e., deduced with the help of bridge laws, to be—not the given (perhaps: historical predecessor) "top" theory T_t—but an

analogous theory T_t^*. Here, the Nagelian agrees that the notion of analogy is very likely to be specific to the theories concerned, but it will be subject to constraints such as (i') and (ii') at the start of Sect. 5.2.

Evidently, causal set theory's overall strategy, of showing Lorentzian manifolds to be approximations to appropriate causal sets, is a case of endeavouring to deduce just such an analogous theory T_t^*. For the idea of analogies between the continuum space and the postulated discrete underpinning has been centre-stage for us, from Sect. 5.2.1's introduction of the Planck scale right through to Sect. 5.7's report of the behaviour of some estimators of continuum quantities such as manifold dimension. These analogies are too clear to be worth rehearsing again *seriatim*.

But on the other hand, as I said in this Section's preamble: causal set theory is still far from having a fully explicit Nagelian reduction of Lorentzian geometry to causal sets, even allowing for deducing an analogous theory T_t^*, rather than the given T_t, i.e., Lorentzian geometry. That lack is no surprise, let alone a defect: for even apart from causal set theory, the formulation of Lorentzian geometry in a form rigorous enough for us to explore the project of deducing it from other theories is in its infancy.[43]

So in the absence of such a formulation, the remaining point that is worth noting in this Chapter is that there are two broad approaches with which one can take causal set theory's prospective reduction to illustrate the Nagelian account. These two approaches can be labelled with slogans or catchphrases, as: either (A): we deduce at the top *t* level what causal set theory is committed to; or (B): we take the top/bottom contrast, *t*/*b*, to be given by the continuum/discrete contrast.

I shall briefly present (A) and (B) in turn; and then remark that so far as I can see, it makes no odds which of these two approaches we adopt.[44]

(A): The idea is, first, to take T_t to be general relativity. Of course, general relativity is really a whole science, not a single theory; so one must be more precise. We have confined ourselves to vacuum; and we have also seen that causal set theory restricts itself to causal sets without closed causal chains, and correspondingly it endeavours to reduce the class of *distinguishing* spacetimes (recall the Hawking-Malament theorem). So we might take T_t to be (a precise formulation of) the theory of distinguishing Lorentzian manifolds. Or we might build in to the specification of T_t a restriction to "unwrinkly" geometries, i.e., to Lorentzian manifolds with curvature only above the Planck scale; (a restriction that, as we have seen, is yet to be made precise). Or, nodding to the semantic conception of theory: we might take T_t to be, not a linguistic formulation, but a (suitably structured) class of models, e.g. the class of distinguishing Lorentzian manifolds; or the class of those that have curvature only above the Planck scale.

Then we take the analogous theory T_t^*—that we aim to deduce from our T_b, i.e., from causal set theory—to be the theory of those causal sets that faithfully embed into the "unwrinkly" manifolds of our T_t.[45] Or, nodding again to the semantic conception: we might take T_t^* to be the (suitably structured) class of those causal sets.

Then the *Hauptvermutung* will be a (precisified) claim to the effect that T_t^* *is* indeed analogous to T_t. So it is an *intra-level* claim about the top *t* level. And this claim is to be justified by a proof: which will presumably in some way exploit, or at least be witnessed by, the good behaviour, reported in Sect. 5.7, of estimators of manifold quantities such as dimension.

So to sum up this approach (A):— Causal set theory T_b is to give a reduction of the geometry of distinguishing spacetimes T_t; with the theory of faithfully embedded causal sets as the analogous theory T_t^*, and the *Hauptvermutung* as the claim of analogy, supported by the good behaviour of estimators of dimension etc.

(B): In this approach, we take the top/bottom contrast, *t*/*b*, to match the continuum/discrete contrast. So we can take T_t exactly as in the first paragraph of (A) above: for example, as (a precise formulation of) the theory of distinguishing Lorentzian manifolds. But unlike (A), all the doctrine about causal sets is to be at the bottom level. So T_b comprises not just causal set theory in general, but also the theory of those causal sets that faithfully embed into the "unwrinkly" manifolds of our T_t. (Or, nodding yet again to the semantic conception: we might take T_b to be the (suitably structured) class of those causal sets.) So the *Hauptvermutung* is now an *inter-level* claim. Furthermore: since on this approach, the top level contains only doctrines about continuum objects, there is on this approach no distinctive analogous top level theory T_t^*. One can, however, think of the completion of a sequence of faithfully embedded causal sets (see Sect. 5.5.2) as the deduction of the continuum limit; and so as a transition from the bottom level to the top.

So to sum up this approach (B):— As on approach (A), causal set theory T_b is to give a reduction of the geometry of distinguishing spacetimes T_t. But the theory of faithfully embedded causal sets is here part of T_b; it is not the analogous top-level theory T_t^*. So the *Hauptvermutung* is an inter-level claim: needing, again, to be proved, and supported by the good behaviour of estimators of dimension etc.

As I said, I think it makes no odds which of these two approaches (A) and (B) we adopt. For on either approach,

the detailed work (mostly in theoretical and mathematical physics, not in philosophy) that needs to be done is the same.

5.8.2 The Prospects for Functionalist Reduction

I turn to functionalist reduction. Achieving a functionalist reduction will surely be a harder task than Nagelian reduction, simply because (as I said in Sect. 5.3.1) it is a logically stronger concept. For it adds the requirement of simultaneous specifications of items by their inter-connected functional roles; indeed at both the bottom and the top levels (and so making the bridge laws mandatory). Thus as I said in Sect. 5.3.2, the question arises: is causal set theory *en route* to providing, not just a Nagelian reduction, but a functionalist reduction, of Lorentzian geometry?

The main thing to stress is that such a reduction will be uphill work. For consider, to begin with, just the top level, about Lorentzian geometry. It is clear from Sect. 5.8.1 that we do not now have a formulation rigorous enough to provide precise functional roles for the concepts of interest, such as manifold-dimension, or length of a timelike geodesic or curvature; or to settle whether they are indeed simultaneously specifiable by their functional roles (i.e., whether the formulation is rich enough to entail such specifications). The subject is just not yet sufficiently developed (see Footnote 43).

Furthermore, even if we had such a formulation, both rigorous and rich enough to entail such specifications, so that the top level's T_t (or T_t^*) was a good candidate for functionalist reduction: nevertheless, the functional roles of the concepts of interest, e.g. manifold-dimension, will of course *not* be exactly filled at the bottom level, i.e., by some feature of causal sets—because of the deep

differences between continuous and discrete spaces. Such differences have been centre-stage throughout this Chapter. But recall in particular, from the start of Sect. 5.7, how defining an estimator on causal sets for a continuum quantity often calls for ingenuity. So obviously we are back at the "impure" case mentioned towards the end of Sect. 5.3.1 (and discussed already by Lewis, e.g. (1970, 432; 443–446)), where a functionalist reduction depends on there being *near-realizers*. That is, one hopes that the functional roles at issue are nearly filled at the bottom level by concepts (quantities), each of which fills its role better than anything else does.

Ascertaining whether this is so will need detailed work, comparing the behaviour of various bottom-level quantities. In particular, in the probabilistic framework of Sect. 5.7, one would need to compare various proposed estimators of each continuum concept or quantity of interest, such as dimension. Such comparisons would of course draw, not just on mathematical work, but also on judgments of analogy as discussed in Sect. 5.8.1. Again, I think it would make no odds which of the approaches (A) and (B) we adopt.

Let me sum up this discussion. Whether we adopt approach (A) or approach (B) of Sect. 5.8.1, a functionalist reduction requires two main results:

(i): In the continuum theory (for example, of all distinguishing Lorentzian manifolds): one needs to argue that dimension, timelike length, curvature etc. are simultaneously specified by their functional roles, i.e., their relations to each other.

(ii): In causal set theory: one needs to argue that the analogous functional roles (of, for example, Sect. 5.7's estimators) provide a simultaneous specification of them, that is sufficiently analogous to (i), so as to give a functionalist reduction. For example: causal set theory's

dimension estimator needs to be near enough the realizer of the functional role of continuum-dimension.

In short, at the bottom level as well as at the top, a functionalist reduction will need detailed work.

So much by way of presenting my prognosis, 'uphill work.' But causal set theory's still having plenty of work to do does *not* mean that one should be sceptical or down-hearted! For as I have reported (Sects. 5.5.2 to 5.7), the *Hauptvermutung* has had a good number of what I called success stories. So causal set theorists have reason to be confident that the detailed work, and its supporting arguments like judgments of analogy, will deliver a reduction. Of course, they can be more confident of achieving a Nagelian reduction, than the logically stronger idea of a functionalist reduction—but should we not try to reach for the stars?[46]

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Footnotes

1 Section 5.2 gives details about reduction. But as to the mnemonics: 't' is my mnemonic for 'top,' 'tangible' or 'tainted,' and 'b' is my mnemonic for 'bottom,' 'basic' or 'better.' It is more common to talk of T_1 and T_2. But this can make it confusing which theory is reduced, i.e., is shown to be a part, and which does the reducing, i.e., is shown to be the whole. And it can be more confusing since physicists often use 'reduce' for the converse relation. That is, their jargon is often that the 'bottom,' basic or better theory *reduces to* the 'top' or tainted one, especially in some limit. For example, they say that special relativistic particle mechanics reduces to Newtonian particle mechanics as $c \to \infty$. I shall adopt the former, philosophers' jargon.

2 Details of this founding paper's proposal are in Sect. 5.5. Another note about jargon: it is common to call a diffeomorphism equivalence class of Lorentzian manifolds (M, g) (M a manifold, g a Lorentzian metric) a *Lorentzian geometry* G, so that one writes e.g. $G = \{(M, g)\}$; or with square brackets for an equivalence class, $G = [(M, g)]$. And I shall adopt this usage when needed, as it sometimes is: after all, the idea of diffeomorphism invariance makes a Lorentzian geometry, rather than a single Lorentzian manifold, the (kind of) object that causal set theory must recover. But 'Lorentzian geometry' has an even more entrenched usage as the name of our mathematical theory of Lorentzian manifolds. So while I of course "aspire" to diffeomorphism invariance, I shall mostly talk of Lorentzian manifolds, causal sets recovering such manifolds etc.

3 For example, as to positive results: even the founding paper addresses how to recover the Einstein field equations (see Bombelli et al., 1987, 523–524); and see the references below, including in Footnotes 33 and 36).

4 As this brief and perhaps contentious summary suggests, my sympathies lie with the syntactic conception; so that my notations T_t, T_b can be read as sets of sentences. I am also sympathetic to the traditional Nagelian account of reduction (for further details, see Sects. 5.2 and 5.3). But I believe that the Chapter's main points, even in the concluding discussion of reduction (Sect. 5.8), will carry over to a semantic conception of theories.

5 So this is a very different use of probability than occurs in probabilistic proposals for causal set dynamics, such as Rideout and Sorkin (2000). There,

one undoubtedly takes the probabilities as irreducible objective chances. But here, all the basic probabilities concern sampling one object i.e., one spacetime point, rather than another. So one answers the usual philosophical question 'Are these probabilities subjective or objective, and in what sense?' with whatever is one's background philosophical view about random sampling. Thus suppose you believe your random number generator, or what-not, that determines which point is selected uses objective chances, while another philosopher thinks it uses subjective probabilities. Then nothing in the philosophy or the physics of causal sets will tilt the debate between you, one way or the other.

6 In particular, it is diffeomorphism-invariant, i.e., independent of which manifold within a diffeomorphism equivalence class one picks. So in the jargon of Footnote 2, it defines a function on Lorentzian *geometries*, not just on Lorentzian manifolds.

7 Or geometry: see Footnote 6.

8 But I think my main points will carry over to the semantic conception; see Sect. 5.1, Footnote 4 and Sect. 5.8.

9 Of course, in order for T_b to be an improvement on T_t, even while reducing it, we would expect there to be other reasons for favouring it over T_t. For this, the obvious, and traditionally emphasised, reason is novel predictions, i.e., that T_b makes confirmed predictions outside the domain of T_t's empirical success. But there are also other reasons (often dubbed 'theoretical virtues'), which are less easily analysed and more controversial than empirical success; such as theoretical unity and simplicity.

10 At least, if we read him with hindsight: see Heilbron (2010, especially Chap. 4) for a magisterial antidote to the temptations of anachronism.

11 And as to being an improvement on T_t (i.e., Galileo's theory), not just reducing it (see Footnote 9): Newton's theory has of course a wealth of confirmed predictions outside the domain of T_t's success, and other theoretical virtues.

12 For defences of the Nagelian account, see Dizadji-Bhamani et al. (2010), Schaffner (2012); my own efforts include Butterfield and Gomes (2023, Sects. 2 and 3) and Butterfield (2014, Sects. 1.2, 4).

13 Of course, there can be no debate over stipulative definitions: the debate is over whether the classifications that ensue from an author's definitions illuminate the details of the theories, and of how they are applied.

14 I developed this view in Butterfield (2011a, 2011b, 2014). But my focus on limiting systems was of course unoriginal: it owed much to discussions by physicists, such as Berry (1994), Kadanoff (2009) and Landsman (2006) and philosophers such as Batterman (2002). More details, and later developments, of (A) and (B), especially including phase transitions in thermal physics, can be found in e.g. Landsman (2013), Lavis et al. (2021), Palacios (2022, especially Chap. 2), and van de Ven (2023).

15 And for the converse relation, causal set theorists often talk of a Lorentzian manifold 'approximating a causal set': a jargon I will also adopt.

16 Hence I wrote 'would be to say'.

17 This paragraph's apparent conflation of the distinction between reduction and supervenience is entirely deliberate. For let us take reduction, following Nagel, as deduction using judiciously chosen definitions of the reduced theory's notions; as in (1) at the start of Sect. 5.2. This implies that supervenience (also known as 'determination' and, among model-theorists, as 'implicit definition') turns out to be generalised reduction, in which the definitions are allowed (but not required) to be infinitely long. For example, it can be an infinite disjunction of the sort philosophers discuss in the multiple-realizability argument; (see Butterfield (2011a, Sect. 4), Butterfield (2011b, Sects. 4.2.3, 5.2.3 and 6.3.4) and Butterfield and Gomes (2023, Sect. 3)). I also argued (in Butterfield, 2011b, under the label '(3:Herring)') that this finite/infinite contrast is, by and large, *not* important, scientifically or even philosophically. A clarification: note that this is a very different finite/infinite contrast from the one at issue in the reconciliation given by (A) and (B) above—which is indeed important.

18 Again, ‘before’ is not really temporal: it just means ‘away from’.

19 Landsman (2013, 382) and van de Ven (2023, 2) generously call this ‘Butterfield’s Principle,’ My (Butterfield, 2011b) had used the label ‘(2: Before)’; and had labelled my scepticism about the scientific and philosophical importance of supervenience, recalled in Footnote 17, ‘(3:Herring)’.

20 I stress that my saying that the focus, or what matters scientifically and philosophically, is the situation *en route* to, i.e., before, the limit—in our case-study: the situation at the Planck length and above—does *not* mean there is no interest in studying the limit. Just as in the cases, e.g. in thermal physics, discussed in the studies in Footnote 14, studying the limit teaches us a lot about the passage towards the limit. We shall see several examples of this from Sect. 5.5.2 onwards.

21 Notice that this idea, and this image, is indifferent to the contrast between reduction and supervenience. It makes no difference whether some, or even all, of the elite minority of quantities whose values define the macrostates can only be defined in terms of the many quantities on the microscopic constituents, by some infinitary construction such as taking a limit. See Footnote 17.

22 Besides, as announced in Sect. 5.1: the idea of approximate intersection will be made precise in terms of probability theory. Indeed, the appropriate functions on the set of causal sets will be random variables that are estimators of continuum quantities like dimension and curvature.

23 This is not say that such an application is easy or common: I am not an imperialist about functionalist reduction. For as I also said in Sect. 5.3.1, the requirement of simultaneous specifications of concepts makes functionalist reduction logically stronger than traditional Nagelian reduction. Of course, this makes it all the more striking that spacetime theories give well-worked out examples.

24 As I mentioned (Sect. 5.1 and the start of Sect. 5.3), philosophers have recently addressed this question. But I disagree with parts of Lam and

Wüthrich (2018, Sect. 4, 2021, Sect. 4). For (like most of the literature about spacetime functionalism) they do not cite Lewis, nor engage with the ideas of simultaneous specification and functionalist reduction. No matter: Huggett and Wüthrich (forthcoming, Sects. 2.4 and 3.3) make amends; and as I mentioned, my remaining disagreement with them can wait till Sect. 5.8.2.

25 For more details, and other examples, see Gomes and Butterfield (2022, 2024).

26 I say 'along the lines' because of course the right-hand-side's use of 'judge simultaneous' needs spelling out.

27 Winnie (1977, especially Sections V, VI) and Goldblatt (1987, Appendix B) are modern expositions emphasising logical rigour. Sklar (1977, 1977) are philosophical discussions, including the idea of a "causal theory of time"—which is of course kin to causal set theory.

28 This example is of course better known for a different purpose than illustrating a derived bridge law, or the idea of functionalist reduction. Namely, as causing trouble for Reichenbach and Grunbaum's doctrine that simultaneity in special relativity is conventional, largely because it seemed to them *not* to be causally definable. (They had in mind the suggestion, 'events are simultaneous iff not causally connectible'; and did not know about Robb's work.) This critique is mostly credited to Malament (1977): who indeed added to their trouble by proving (his Proposition 2) that the orthodox, textbook notion of simultaneity (relative to an inertial worldline L) is the unique non-trivial equivalence relation on Minkowski spacetime that is (even just implicitly) definable from causal connectibility (or equivalently: *After*), together with membership of L.

29 This is a very advanced example, which is not as well known among philosophers of space and time as it should be. But for introductory details about the general issue, both historical and philosophical, see e.g. Van Bendegem (2019) and Kragh and Carazza (1994).

30 Clearly, this selection will need to be made uniformly, in some sense, across the manifold, so as to avoid lacunae: that is, so as to not omit regions whose

contents encode features of the continuous space that are not encoded in other regions; e.g. a region that contains a topological hole.

31 Agreed, not all structure is defined at arbitrarily small scales: some features are 'global,' e.g. the existence of a topological hole. Cf. Footnote 30.

32 For brevity, I have stated desideratum (B) in terms of quantities, which one thinks of as number-valued functions of the state. But of course, a similar desideratum applies to mathematical structures that are not naturally thought of as a number-valued function: such as being a topological space, or being Hausdorff. How to make precise 'no or small variation from the values and structures described by the continuous space (or theory) in the regime of its empirical success' will of course be a subtle affair, and depend on the quantity or structure concerned. Recall the discussion of *approximate supervenience* in Sect. 5.2.1.

33 Reviews including the treatments which I set aside, as well as what I *do* need, include Brightwell and Luczak (2015), Sorkin (1991, 1991a, 2005) and Surya (2019). This last is especially helpful, being detailed as well as recent. It treats matter and radiation in its Sect. 5, and dynamics (both classical and quantum) in its Sect. 6 (with kinematical prerequisites in its Sects. 4.4 and 4.5). Carlip, Carlip and Surya (2023) is a fine recent article about quantum dynamics. I will also draw on Sects. 3.3 and 3.4 of Butterfield and Dowker (2023), whose Sect. 6 reviews dynamics. For a philosophical exposition complementary to mine: Huggett and Wüthrich (forthcoming) present this basic "classical vacuum kinematics" (Chaps. 2.2 and 3.2). Also like me (in Sects. 5.7 and 5.8), they discuss causal set theory's "replacements" for manifold dimension, and for timelike geodesic distance (their Chap. 2.3.3 and 2.3.5 respectively). But they also go beyond my remit: they discuss (mostly classical) dynamics, how to accommodate the idea of temporal becoming, the idea of effective Lorentz symmetry, and the subtle role of non-locality in dynamics (Chaps. 3.2.2, 3.4 and 3.5). Also, Forgione (this volume) discusses causal set theory's relation to philosophers' proposals for a "causal theory of time"; cf. Footnote 27.

34 I stress that despite the name, the success of causal set theory by no means depends on the *Hauptvermutung* being proven. For it sets aside causal set theory's proposals about matter, radiation and quantum theory: and it could turn out that the successful recovery of even vacuum classical general relativity somehow depends on the details of those proposals—so that the purely geometric formulation of the *Hauptvermutung* dealt with here falls by the

wayside, as not needed in the overall endeavour of recovering (within error-bars) all the empirical successes of general relativity, which are of course mostly *non*-vacuum.

35 See in addition to the main text to follow: Bombelli et al. (1987, 523), and Sorkin (1991a, 7, 2005, 11).

36 For a more detailed review, see Butterfield and Dowker (2023, Sects. 3.3 and 3.4). In particular, Sect. 5.5.2's strategy is not the only geometric one. For I here set aside the effort to try and define a metric on the set of Lorentzian manifolds, by adapting the Gromov-Hausdorff distance between metric spaces to metrics of Lorentzian signature: the hope being that two manifolds that are metrically close are in some appropriate sense approximately isometric. See Bombelli and Noldus (2004) and its reference 6 and 7; Belot (2011, Chap. 1.3, and Appendix B) is a superb introduction for philosophers to ideas and results about Gromov-Hausdorff distance.

37 I say '*primary* physical significance,' because, as I admitted in Sect. 5.2.1: theorems and constructions about the continuum limit are indeed often illuminating, and even physically significant. But I think that significance lies in what information they imply about the regime before the limit: in other words, a continuum limit is evidence of a continuum *approximation* "nearby".

38 This summary makes clear, as Bombelli emphasises, that the proposal does not require any identification of points between manifolds, and is thus diffeomorphism invariant. That is: the distance function is well-defined on isometry classes of Lorentzian manifolds, i.e., on Lorentzian geometries in the sense of Footnote 2. In other words, we can replace my words 'sample ... in each of the manifolds' above, by 'sample ... in one representative of each of the geometries'.

39 An apparently equivalent formulation is given in Brightwell and Luczak's review (2015, 9). They use the idea, for a region U of a Lorentzian manifold of finite volume, and a finite causal set C, of a *density* $t(C; U)$, which is defined as the probability that a Poisson sampling of $|C|$ elements from U yields a causal set isomorphic to C. Then their formulation is: if U and V are regions of Lorentzian manifolds of finite volume, such that there is no measure-preserving diffeomorphism between the two, then there is some finite causal set C such that $t(C;U) \neq t(C;V)$.

40 Of course, Robb's causal theory of Minkowski spacetime required similar ingenuity, as the formulation of bridge laws in reduction usually does (see Sect. 5.3.2). But unlike Robb, we here cannot exploit the global and linear structure of the spacetime we wish to recover.

41 See Sect. 5.2.1 and Footnote 37.

42 As I said in Sect. 5.1, I will not comment in detail on recent philosophical discussions by Lam, Huggett and Wüthrich, since for the most part I agree with them. In short, we agree that causal set theory is *en route* to reduction: promising but not there yet. But Sect. 5.8.2 will record one disagreement. Also: as I said in Footnote 4 and the start of Sect. 5.2: I focus on the syntactic conception of a theory, and so on Nagelian reduction. But I believe my main points will carry over to a semantic conception of theories.

43 See Andreka et al. (2007, especially Sect. 3.6).

44 For further discussion, especially of levels and of kinds of emergence, see Oriti (2023, especially Sects. 1.2 and 4).

45 I note again that characterising which causal sets can be thus faithfully embedded is a task for causal set theory's dynamics: agreed, a major task in view of the vastly many causal sets that will not thus embed (Kleitman & Rothschild, 1975). But I have set this aside; cf. again the references in Footnote 33, e.g. Carlip and Surya (2023).

46 I should briefly compare my discussion with Huggett and Wüthrich (forthcoming) discussion, especially Sects. 2.4, 15–16, and 3.5.4. The main point is that, broadly speaking, we agree. Here, I am grateful to Nick Huggett for clarifying correspondence. They point out that, as we have seen, causal set theory admits many causal sets that do not correspond to a classical spacetime. So they stress, and I agree, that the theory thus needs to earn physical significance by linking its posited causal sets to such spacetimes, at least to the extent of recovering (at least much of) our evidence for our established physical theories postulating such spacetimes. Here, there is a contrast with

the mind-body example of Sect. 5.3.1: where the neurophysiological bottom-level is not considered problematic, or as needing to earn physical significance. But they allow that discreteness is itself no bar to earning physical significance. More positively, they recall (2023, Sect. 3.5.4) that causal set theory *can* claim observational evidence of a striking kind, which furthermore fits the philosophical stereotype of a novel prediction turning out to be true (see my Footnotes 9 and 11). Namely, it predicts that the cosmological constant has a small but positive value: this prediction was first made by Sorkin in (Sorkin, 1991, 22), some twenty years before observational confirmation. With all this, I of course concur.

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6. Causation as Constraints in Causal Set Theory

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Abstract
Many approaches to quantum gravity—the theory that should account for quantum and gravitational phenomena under the same theoretical umbrella—seem to point at some form of spacetime emergence, i.e., the fact that spacetime is not a fundamental entity of our physical world. This tenet has sparked many philosophical discussions: from the so-called 'empirical incoherence problem' to different accounts of emergence and mechanisms thereof. In this contribution, I focus on the partial order relation of causal set theory and argue that causation can be characterized as an a-temporal constraint over the kinematic space defined by the theory. The relation constrains the growth of a new element/event with respect to the other elements/events of a given set. Therefrom, the flow of time emerges from the collection of the possible growths of the given set, where each possibility is

characterized by a classical probability assigned by the dynamics of the theory.

6.1 Introduction: Fundamental Causal Relations

Perhaps, one of the most challenging enterprises in contemporary theoretical physics is the development of a unified theory of gravitation and quantum mechanics: a theory of quantum gravity. The theory is meant to solve some of the problems that still affect two of the most well-verified theories of modern science: the general theory of relativity and quantum field theory. Both theories deliver, in their own domains of applicability, some astounding experimental predictions, and yet they both fail to provide a comprehensive account of some fundamental questions about the beginning of the universe, black holes, quantum effects of gravity, and many others. To make things worse, general relativity (GR) and quantum field theory (QFT) seem to be fundamentally incompatible with one another, to the point that even a brief (and incomplete) overview is enough to show the conceptual (and mathematical) difficulties of the unification task. General relativity is a deterministic theory about spacetime, gravity, and matter. As Wheeler's motto famously says: *matter tells spacetime how to curve, and spacetime tells matter how to move*. Spacetime is treated as a dynamical and continuous entity that admits no preferred spatial or temporal directions, and the interactions of the many physical systems that live in such spacetime are local. Quantum field theory, on the other hand, studies the dynamical fields that 'live' on a non-dynamical background. The theory is characterized by the uncertainty principle, and thus by the lack of complete localizability. Finally, quantum mechanics is fundamentally probabilistic and all interactions are quantized.

Many physicists have tried to combine these apparently incompatible domains, leading to the proliferation of different approaches and theories of quantum gravity. Despite the many differences of these approaches, one common feature seems to be that spacetime ought to be treated as an entity that emerges from the interaction of some fundamental entities. This has sparked a prolific discussion among physicists and philosophers about issues such as: the definition of emergence, how to recover the manifest spacetime from non-spatiotemporal entities, how to experimentally probe a theory of quantum gravity, and many others.

With the present contribution, I focus on characterizing the fundamental relation of causal set theory as a constraint over the possible configurations admitted by the theory. More specifically, Sect. 6.2 offers a brief presentation of the main axioms of the theory and its kinematics. I shall emphasize the relationship between those causal sets that represent possible classical configurations of our universe, and the causal tree that represents the multiplicity of possible growths. Section 6.3 discusses the role of the time parameter in causal set theory as bookkeeping device for the position of a given event relative to the other events of the causal set. Then, I maintain that causal relations in CST are more fundamental than their spatiotemporal counterparts. This latter claim opens up the possibility for interpreting the partial order relation as a constraint principle. Section 6.4 begins with a brief overview of some common accounts of causation in philosophy of science. Then, starting from a pluralistic account, I will characterize the relation of causation in CST (the partial order relation) as a constraint over the possible growths of all possible configurations of the casual sets. Section 6.5 provides some concluding remarks.

6.2 Causal Set Theory

With this section I offer a review of some of the fundamental tenets and results of CST, but deeper reviews and discussions can be found in: Sorkin (2005), Wüthrich and Huggett (2020), Surya (2019), Henson (2009), Wallden (2013), Dowker (2005), and others.

The central idea of Causal Set Theory (CST) is that spacetime is fundamentally discrete and its structure is that of locally finite partially ordered sets that represent possible kinematic configurations of the universe. The theory follows the sum-over-histories approach for which, starting from a space of possible histories, one assigns a measure to the individual trajectories to calculate a final amplitude as sum-over individual histories. In causal set theory, the space of possible histories consists of discrete structures to which Lorentzian manifolds are only an approximation, that is, the causal sets. There are typically two formulations of the theory in the literature and they are distinguished by the properties of their causal relation. The *reflexive* formulation, which is also the least common one, makes use of a causal relation that is reflexive, antisymmetric and transitive.

The more common version of causal set theory makes use of an irreflexive relation which does not allow for instances of self-causation. Since the latter version is more common, I shall use it for the rest of this contribution. The theory is defined by the following six axioms (Dribus, 2017, 151):

1. Binary Axiom: Classical spacetime may be modeled via a set $\mathcal{C}$, called a causal set, whose elements represent spacetime events, together with a binary relation $\prec_{CS}$ on $\mathcal{C}$, called the causal set relation, whose elements represent causal relationships between individual pairs of spacetime events.

2. Measure Axiom: $\mathcal{C}$ is equipped with a discrete measure μ_{cs}, called the causal set measure, which assigns to each subset of $\mathcal{C}$ a volume equal to its number of elements in fundamental units, up to Poisson-type fluctuations

3. Countability: $\mathcal{C}$ is countable.

4. Transitivity: Given three elements *x*, *y*, and *z* in $\mathcal{C}$, if $x \prec_{cs} y \prec_{cs} z$, then also $x \prec_{CS} z$.

5. Interval Finiteness (or) Local Finiteness: For every pair of elements *x* and *z* in $\mathcal{C}$, the open interval

$$\langle\langle x, z\rangle\rangle := \{y \in \mathcal{C} | x \prec_{cs} y \prec_{cs} z\} \tag{6.1}$$

has finite cardinality.

6. Irreflexivity: Elements of $\mathcal{C}$ are not self-related with respect to $\prec_{cs}$ i.e., $x \not\prec_{cs} x$

An example of how causal set theory models a universe composed of three elements is represented in Fig. 6.1, where each node represents a given causal set with its corresponding events and relations.[1]

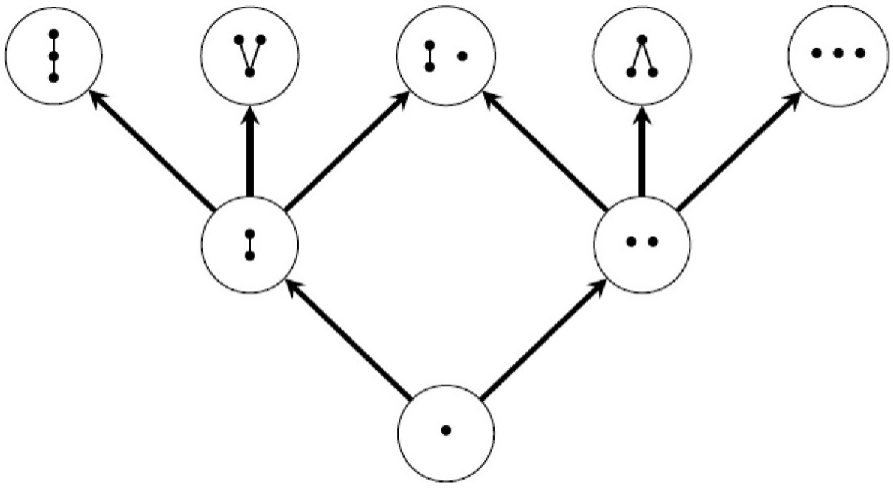


Fig. 6.1 A simplified causal set tree up to three elements. Cf. also Dowker et al. (2020)

From the simplified three-elements model, one individuates two distinct 'levels' in causal set theory: one the one hand we have the individual nodes that represent possible configurations of classical universes with one, two or three elements:

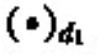
2]6

$$S_{cs} : \left\{ \bullet, \vdots, \bullet\bullet, \bullet\bullet\bullet, \vdots\ \bullet, \Lambda, \mathrm{V}, \vdots \right\} \tag{6.2}$$

On the other hand, the tree-structure represents the multiplicity of possible growths from one universe-configuration to another. It corresponds to the quantum-level of the theory that represents a kinematic scheme S_{cs} of sequential possible growths.[2] As already pointed out by Dribus (2017), the two levels of causal set theory seem to question the popular idea that classical mechanics is less fundamental than quantum mechanics, while here it seems that the classical nodes are more fundamental than the quantum kinematic scheme. Here, the analogy with the sum-over-histories can help us clarify the conundrum. Indeed, in the sum-over-histories approach (alternatively, path integrals), the individual trajectories can be considered as 'classical', for they are individuated by a classical action functional. The quantum properties emerge as soon as we consider the total ensemble of possible trajectories, in that each trajectory is assigned a probability amplitude which sums with all the other possible paths.

Notably, terms such as 'reduction' and 'fundamentality' become somewhat tricky here. Even without diving deep into the literature, consider the following two readings: on the one hand it is intuitive to consider quantum mechanics as more fundamental than classical mechanics—for example because the former operates at scales smaller than the latter. On the other hand, quantum mechanics

dictates that all possibilities count—for example, all trajectories in the path integral—and it is only at the classical limit that such a multitude of possibilities reduces to one. This is especially evident if we look at how, at the limit $\hbar \to 0$, the many possible trajectories of quantum mechanics constructively interfere around the classical trajectory, thereby reducing the path integral to the least action principle.[3] Therefore, we shall rely on a temporary workaround to the problem of characterizing terms such as 'reduction' and 'fundamentality': we shall more timidly say that quantum mechanics and classical mechanics represent different structures and that while the former interests a specific physically possible scenario, the latter dictates that all possible physical scenarios count.

I will say more on the dynamics of the theory below, but for now it is enough to say that it provides a rule for the probabilities of the co-relative histories (or) transitions from one node to another. The most common approach is named Classical Sequential Growth (Rideout & Sorkin, 1999) and it assigns classical probabilities to the possible transitions from one causal set to another in the kinematic scheme. For example, in the simplified tree in Fig. 6.2, the transition from empty set $(\emptyset)_{d_0}$ to  is $P(d_1 \to d_2) = 1$ and the probability of the realization of causal set is $P(d_t) = 1/3$. Notably, The probabilities assigned by CSG are still classical, which testifies the incompleteness of causal set theory as a full-fledged theory of quantum gravity.

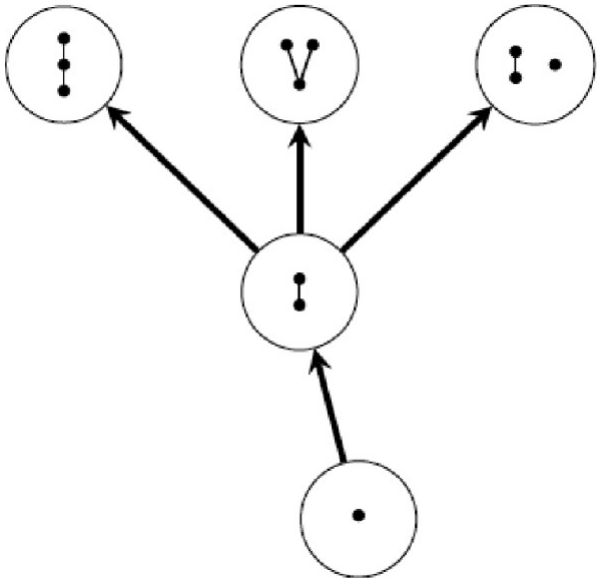


Fig. 6.2 Simplified model of a causal set tree

6.2.1 Why Causal Set?

Thus far, we have seen some of the general features of the theory, but I have not really mentioned why the use of Hesse diagrams and causal sets can be useful to recover relativistic spacetime. The central theorem of the causal set programme is the Hawking-Malament theorem (Hawking et al., 1976; Malament, 1977)—also referred to as the Metric Recovery Theorem (MRT) by Dribus (2013, 2017). Broadly speaking, the theorem proves that: "the causal structure of relativistic spacetime determines the corresponding metric structure up to smooth conformal isometry" (Dribus, 2017, 91). Here, *relativistic causal structure* means that the future timelike and past timelike directions are not dependent on the choice of frame of reference—and thus causal influences propagate always from causes to effects.[4]

Physically, the theorem states that relativistic spacetime can be reconstructed starting from the description of all the causal relations between the events populating spacetime. In this sense, Wheeler's motto: *space tells matter how to move, and matter tells space how to curve* emphasizes the prescriptive character of geometry and

matter, which is captured by the theory of general relativity. But, instead of prescribing how space bends and matter moves, one can provide a description of all the causal relations between the events populating spacetime. The metric recovery theorem states that prescriptions and descriptions convey the same information. However, this latter statement is imprecise and the theorem adds the condition *up to smooth conformal isometry*: a tricky requirement that requires some unpacking.

First, we shall begin with the definition of *conformal equivalence* as, for example, stated in Dribus (2017, 87): "two pseudo-Riemannian metrics g and g' on a smooth manifold X are called conformally equivalent if there exist a smooth positive function $\omega : X \to \mathbb{R}$ called the conformal factor, such that $g'(v, w) = \omega(x)^2 g_x(v, w)$ for every point $x \in X$ and every pair of tangent vectors $v, w \in T_x X$." In layman's terms, two structures are conformally equivalent as long as there is a function between them that (locally) preserves angles, while the measuring rods (lengths) might vary. A conformal isometry is an even stricter condition, since it imposes the existence of a diffeomorphism between two metrics such that the metric $f * g'$ on X is conformally equivalent to g on X. Therefore, the problem of reconstructing relativistic geometry from its causal structure is the lack of the conformal factor (that is: scale data) on the relativistic manifold. This is intrinsic to GR, since the diffeomorphism invariance, for example, makes it impossible to obtain those scale data from some other ambient scale (e.g.: an embedding manifold).

The workaround to the missing conformal factor is the axiom of local finiteness for which all ordered intervals in the causal set have finite cardinality, thereby corresponding to a finite cut-off which is interpreted as a measure of volume. Therefrom the slogan coined by Sorkin: *Order plus number equals geometry*. While the term *order*

stands for the binary relation on the events of a causal set, the term *number* stands for the natural scale—that is, the local finiteness axiom associated with a measure of volume μ_{CS} which consists of counting the elements of a given causal set *C*.[5] Finally, the term *geometry* stands for relativistic spacetime. Bombelli (1987, 522) suggested that the observables of causal set theory should correspond to topological and metric observables, and that the latter should be expressed in terms of a deeper notion of order:

> In this view, volume is number and macroscopic causality reflects a deeper notion of order in terms of which all the 'geometrical' structures of space-time must find their ultimate expression.

Continuum based geometry is then recovered *via* smoothing at large scales the structure of causal set *C* and binary relation $\prec_{CS}$. The operation is warranted by the notion of *embedding*, which consists of an injective map $\phi : C \to (M, g)$ from a causal set to a pseudo-Riemannian manifold such that given $x, y \in C$:

$$x \prec_{CS} y \longleftrightarrow \phi(x) \prec_{M} \phi(y) \tag{6.3}$$

But, not all causal sets can be embedded into a spacetime (*M*, *g*) and, even if they could, this does not guarantee that a given spacetime is approximated by a given causal set. One shall require that *number* approximates *spacetime volume*, a stricter condition called *faithful embedding*: "every finite spacetime volume *V* is represented by a finite number of elements $n \approx \rho_C V$ in the causal set", where $\rho_C = V_C^{-1}$ (Surya, 2019, 14). In addition, to recover relativistic spacetime one needs to ensure the covariance of the distribution of the elements of the causal set, and this is obtained by inducing a random sprinkling of elements

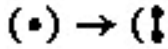
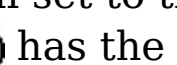
(Poisson process) on pseudo-Riemannian manifolds (Surya, 2019, 16)[6]:

> We say that a causal set C is approximated by a spacetime (M, g) if C can be obtained from (M, g) via a high probability Poisson sprinkling. Conversely, for every $C \in \mathcal{C}(M, \rho_C)$ there is a natural embedding map
>
> $$\phi : C \to M$$
>
> where $\mathcal{C}$ is an ensemble of causal sets obtained from the imposition of a partial order on the elements of the Poisson sprinkling. It remains to determine how a manifold-like causal structure can uniquely determine large-scale manifolds. The uniqueness of the continuum approximation is warranted by a conjecture, the *Hauptvermutung* (fundamental conjecture of causal set theory):
>
> ***The Hauptvermutung of CST:*** C can be faithfully embedded at density ρ_C into two distinct spacetimes, (M, g) and (M', g') iff they are approximately isometric (Surya, 2019, 19).

The conjecture is not proven yet, but some work in this direction has been put forward by, among others: (Bombelli, 2000; Noldus, 2004; Bombelli et al., 2012). Finally, a few words on the most common dynamical approach to causal set theory: the causal sequential growth (CSG) by Rideout and Sorkin (1999). According to the CSG exploratory model, a causal set is built via evolutionary steps i.e., by adding one element to the set at each transition. Each transition (co-relative history) consists of a morphism between two causal sets, thereby making spacetime a dynamical entity that evolves following a co-relative history. As it was mentioned earlier, the kinematic scheme of the theory describes the potential evolutionary steps of a given causal set, while the dynamics (CSG)

supplies with (classical) probabilities that weight each possible transition.

6.3 The Emergence of Time

The presentation of the theory, thus far, has left the notions of causation and time evolution somewhat blurry. Indeed, the individual causal sets do not track a proper time evolution in that they are part of the kinematic scheme. The causal relation of CST does not correspond to a strictly forward in time relation, but this should not be much of a surprise since the relativistic nature of spacetime admits space-like separated events.[7] For example, while the sequence  might suggest a passage of time from one causal set to the other, the co-relative history  has the same probabilities to occur. The partial order relation of causal set theory is hardly a causal relation in a traditional sense, and that is because causal sets are not enough to relate each cause to the corresponding effect. The point was raised by Wüthrich and Huggett (2020) and it is justified by the fact that not all timelike or null relations have causal efficacy: “given an event, we take neither all events in its past lightcone to be its causes, nor all events in its future lightcone to be its effects” (Wüthrich & Huggett, 2020, 13). Again, this validates the idea that causal relations in causal set theory differ substantially from their everyday counterpart: “the causal relations of causal set theory of course differ from that attributed to events in our ordinary lives. Nevertheless, given the tight relationship between the fundamental relation of causal set theory and the causal structure of relativistic spacetimes, we take it to be legitimate to dub this relation a ‘causal relation”’ (Wüthrich & Huggett, 2020, 13).

One could suggest that an external time parameter can be associated with subsequent causal sets, thereby recovering the idea of a flow of time. This might be especially tempting in CSG, since each transition corresponds to the birth of a new element in the causal set. The partial order relation, as well as the so-called condition of internal temporality impose that if we call 'parent' and 'child' two elements of a causal set, then no 'child' element can be born before the 'parent.' In other words: "[…] each element is born either to future of, or unrelated to, all existing elements; that is, no element can arise to the past of an existing element" (Rideout & Sorkin, 1999, 19). We could interpret this process of progressive accretion of a causal set as the manifestation of the 'phenomenological time', and we could do so by imposing a label to each of the subsequent causal sets. To each causal history of the universe—that is, to each sequence of subsequent causal sets—we would have a unique labeling, that is, a sequence of natural numbers that externally tracks the succession of causal sets. However, the order of growth of two events that are spacelike with respect to one another is pure gauge, and thus carries no physical meaning. In addition, different causal histories leading to the same causal set are also gauge invariant, and that is because: "We want the physics to be independent of labeling, so different paths in $\mathcal{P}$ leading to the same [causal set] should be regarded as representing the same (partial) universe, the distinction between them being pure gauge" (Rideout & Sorkin, 1999, 5).

But then, if the time order in the causal tree is gauge invariant, how should we recover a more traditional phenomenological passage of time? For example, Dowker (2014) emphasizes how in general relativity each physical system is locally associated with a world-line that is causally ordered. Then, in causal set theory: "[a] sequential growth model is a model of a physical world which becomes

in a manner compatible with the lack of a physical global time" Dowker (2014, 8). In this sense, causal relations constitute the local ordering of events in a given causal history, and time plays the role of an independent (and un-physical) parameter used to keep track of the different positions of the events with respect to each other within the causal set.

The physical passage of time, as mentioned earlier, should emerge as the resulting effect of the stochastic birthing of successive events. But, because of the invariance under label transformation, we would need a (yet unspecified) process of cancellation of the many gauge invariant causal histories to give rise to an actual history of the universe at the classical limit. In this sense, the analogy with the sum-over-histories approach becomes salient, since, there, a single classical history emerges at the classical limit from the mutual cancellation of the many possible histories of the quantum ensemble. Alas, for the analogy to work, we need a proper quantum dynamics for causal set theory that would dictate the process of cancellation of the causal histories. What remains is that causal set theory does not admit an external time that tracks the evolution of a causal set and, independently of how such a phenomenological time will be recovered, it is safe to conclude that time is an emergent property of the theory.

6.3.1 Causal Theory of Spacetime and Fundamental Causation

If we accept that the passage of time is an emergent property in causal set theory, then we might be able to decouple the partial order relation (causal relation) from the notion of time. The issue is not new in the literature, as, for example, Russell and Slater (2022, 381) expressed it in terms of reduction and fundamentality: "can time be

derived from causality, or must we retain temporal order as fundamental, and distinguish cause and effect as the earlier and later terms in a causal relation?" An affirmative answer is defended by the causal theory of spacetime (CTS), which maintains that the structure of spacetime can be recovered from causal structures. Again, the idea is supported by Hawking and Malament's causal metric hypothesis, but also by authors such as H. Reichenbach and B.V. Fraassen. Most recently, Baron and Le Bihan (2023) have revitalized the debate and defended a version of the causal theory of spacetime for which spacetime relations are grounded on causal relations. Baron and Le Bihan (2023) present two approaches to the causal theory of spacetime: the identity theory and the non-identity theory. The former comes in two flavors: strong and weak, which are distinguished based on whether spacetime-relations are characterized as causal relations a priori or a posteriori respectively.

The a priori approach seeks to define spatiotemporal relations as causal relations, but Baron and Le Bihan (2023) object that we can imagine a world in which spacetime structure exists and there are no causal relations. This scenario implies that the universe, or a subset thereof, would be entirely idle with no dynamics. Conversely, we can also imagine "worlds with causation that do not feature relativistic spacetimes, where space and time are different manifolds (as in Newtonian worlds). These worlds may be physically impossible or even metaphysically impossible, but they are certainly not conceptually incoherent" (Baron & Le Bihan, 2023, 3). They conclude: "[a]ssuming that, if *a* can be conceived of without *b*, then the identity of *a* and *b* is not a priori, there does not seem to be a conceptual link between causation and relativistic spacetime of the right kind" (*ibid*).

Another objection, originally raised by Smart (1969) and recalled by Baron and Le Bihan (2023) is that we should

not identify clear terms with terms that are less clear: “To elucidate the concept of space-time in terms of the concept of causal connectedness seems to be to elucidate the comparatively clear by reference to the comparatively unclear” (Smart, 1969, 394). Surely, reply Baron and Le Bihan (2023), the past decades have brought more clarity to the notion of cause, especially in philosophy of science. However: “[w]hile we have developed theories of causation that have some level of precision, what we take to be the most precise of these—the interventionist account coupled to the structural equation framework—typically foregoes any reductive ambitions and takes causation to be an unanalysed primitive. Arguably, causation is still less well understood than spacetime” (Baron & Le Bihan, 2023, 3).

Perhaps, an even more precise notion of causation could be based on the idea of functional dependence, especially with respect to the use of hyperbolic partial differential equations as best representations of Humean causation (Smith, 2000). Since it is not my intention here to discuss new possible definition of causation in science, I shall simply accept that the notion of spacetime is better defined than that of causation—as also supported by both Baron and Smart. With respect to the weak approach to the non-identity theory, the main objection also comes from Smart (1969, 394): “It is difficult to see how the causal theory of time is applicable to theories which allow for the existence of events which are neither causes nor effects of other events.” The objection, which is dubbed by Baron and Le Bihan (2023) the ‘problem of causal indolence’, is indeed severe for it applies to both the strong and weak approach. Because of the identification of spacetime relations with causal relations: “any spatiotemporal relation must be a causal connection, which means that there cannot be any entities that are causally idle so long as they bear spatiotemporal relations to other entities (which they must if they are located in spacetime)” (Baron & Le Bihan, 2023,

4). The causal indolence objection consists of three different sub-problems: (1) there might be spacetime regions that are free of matter and energy, (2) the existence of timelike connected events that are causally disconnected, and (3) the existence of spacelike events that would need superluminal signals to be causally connected.

The first sub-problem resembles in kind the problem of the strong approach to the identity theory, which I have addressed above. The second sub-problem, is not clear in that there cannot be timelike connected events that are not causally connected. Everything that is in the past-lightcone of a given system constitutes a cause, independently of how weak that connection might be. Perhaps the strongest case for such an argument is to conceive of the Big Bang as a universal common cause for the entire universe. Alternatively, we can consider the formation of our sun as something in our past lightcone and thus as a cause of my writing this paper. With respect to the third problem—that is, the existence of spacelike separated events requiring faster-than-light signals to be causally connected—Grünbaum (1973) already suggested to trade causal connections with causal connectability. However, the trade-off is not a viable solution since, when it comes to spacelike events, a causal connection is not even a possibility unless we reject special relativity. Therefore, suggest Baron and Le Bihan (2023), one needs to look at another way to address the problem of causal indolence.

The suggested solution is to give up on the identity approach and characterize the relationship between spacetime and causal relations as an ontological dependence. The suggestion is that spacetime relations are grounded in causal relations, thereby making the latter more fundamental than the former: “our view is that spatiotemporal relations are grounded in a pattern of more fundamental causal relations between events” (Baron & Le Bihan, 2023, 10). This would constitute a solution to the

problem of causal indolence because spacelike connections between two events would be grounded in the absence of a causal relation: “in particular, if two physical events are not linked by a fundamental causal relation, then that grounds a spacelike connection at the spatiotemporal level” (Baron & Le Bihan, 2023, 11). The account is further refined by adding that being part of a causal structure is a necessary requirement for any spatiotemporal connection, in such a way that all events in a causal set would be in some causal connection with some other events, thereby avoiding isolated elements of the set.[8] One more addition to the non-identity approach is that the total causal structure of a given causal set is rule-governed, that is, the physical laws dictate both what events are causally connected, and also what events can be and cannot be causally connected.

In sum, the non-identity approach establishes a relation of grounding between spacetime and causal relations in causal set theory. Instead of saying that spacetime relations can be expressed in terms of causal relations, Baron and Le Bihan (2023) set forth a form of ontological dependence such that causal relations become more fundamental than spatiotemporal ones. Then, the explanation of the causal connectability between events is due to the total causal structure and to the rules that dictate the possible causal connections within such a structure. However, the total causal structure, as seemingly interpreted by Baron and Le Bihan (2023), corresponds to the causal configuration of all events of the universe, i.e., to a causal set. In addition, the introduction of physics laws that dictate what events can be causally connected adds one additional layer to the ontology of causal set theory, and brings about the difficulty of explaining what such rules would consist of.

It remains that causal relations ought to be more fundamental than spacetime relations, but this conclusion

is independent from the use of the non-identity theory and becomes evident from more general considerations. For example, the fact that the structure of relativistic spacetime can be expressed in terms of causal relations (up to a conformal factor) is but one of the pillars on which causal set theory stands. The fundamental discreteness of the theory is not required for the validity of the metric recovery theorem, and the use of the counting measure to define a natural scale for volume is a strategy that overcomes the *up to conformal factor* condition. What follows is that the theory, by identifying discreteness as one of its fundamental axioms, naturally requires a mechanism for recovering the continuum of relativistic spacetime.[9] It is then natural to assume that the fundamental objects and relations of the theory should be more fundamental than the continuum spacetime relations proper of relativity theories. In other words: if discreteness and partial ordering give rise to spacetime relations, it is only natural that the partial order relations and discrete entities should be more fundamental than the relativistic spacetime relations.

6.4 Causation as Constraints

Thus far, along with Baron and Le Bihan (2023), I argued that causal relations are more fundamental than spacetime relations. Yet, this does not help us clarify the former without any use of the latter. The problem was already mentioned by Smart (1969), in that we should not be explaining a clear concept starting from a less clear one. One possibility is to provide a clear account of causation in physics to explain well-defined spatiotemporal relations with well-defined causal relations. Completing the task is no small feat, and many philosophers have already tried to offer a clear notion of causation (in both physics and other

disciplines). Here, without a pretense of completeness, I will review some of such attempts with a special focus on the interventionist account.

Regularity theories of causation reduce causation to instances of specific pattern of succession (this is in contrast to notions of causal efficacy and causal power). Some fundamental principles of the regularity theories are: the constant conjunction of the same types of events, the contiguity in time and space, the asymmetry between causes and effects. The central idea is that a given event *A* that is lawfully followed by an event *B* can be considered as the cause of *B*. One of the objections to this cluster of theories is that modern science is based on repeatability, and that a single instance of one event following another should not necessarily constitute a causal connection. One could respond that any lawful connection implies that causal connections require repeatability. Yet, this might constitute too strong of a constraint since it remains that the correlation between two events does not always constitute an instance of causation. A possible solution would be to provide an underlying mechanism that explains the relation of causation between the two events, but such mechanisms are not always possible—for example, the speed of light is a causal constraint that does not have an underlying mechanism.

Another approach to causation is represented by the counterfactual theories, originally developed by Lewis (1973). These theories analyze causal relations in terms of counterfactuals of the form: 'if I were you, I would accept the job.' An objection to such theories is that they do not always distinguish counterfactuals from causal relations, as made evident by the example above. The fact that *I am not you* can hardly count as a cause for not accepting the job. The process account of causation focuses on the perduration and progression of phenomena, rather than on

their instantaneous occurrence. Dowe (2000), Salmon (1998), and others, tie the notion of causation to a process of transfer of a conserved quantity between two systems. For example, the transition of a ball through a gaseous medium is a causal process in that we could easily mark the ball with, say, a Sharpie, and observe that the mark is transmitted throughout the entire process. Further refinements of the theory will individuate as markers some conserved physical quantities such as momentum or energy (see, for example: Dowe (2000); Salmon (1997). However, the account seems to fail at distinguishing which events or factors constitute a cause, in that the transmission of a marker does not guarantee that the marker is also a cause for a given event. For example, Woodward (2005, 357) points out that "the feature that makes a process causal (transmission of some conserved quantity or other) tells us nothing about which features of the process are causally or explanatorily relevant to the outcome we want to explain." In addition, the account seems to fail at explaining the lack of those marks as a cause for a given event. For example, statements such as: *I killed my plants by not watering them* are considered cases of causation by omission, which express causation without the transmission of any mark (see: Beebee (2004)).

The interventionist account proposed by Woodward (2005) suggests that the proper way of characterizing causation is in terms of manipulations of some variables within a given causal structure. Causes are factors that, when intervened upon, produce a change in a system that was otherwise unexpected. The view has much in common with the counterfactual account, yet it is restricted to specific classes of phenomena, thereby avoiding some of the vagueness problems of the counterfactual theory. The interventionist approach treats causation as a primitive where: X is a total cause of Y if and only if under some

intervention on X there is an associated change in Y. The definition makes no use of spatiotemporal terms, and it seemingly fits well with the needs of causal set theory. Furthermore, The concept of changing a variable within the system can be interpreted modally, so that interventions need not be taking place in spacetime—this is because the variation of the values of X and Y can be intended as being within the space of possibilities defined by the theory. In addition, the manipulation of one of the variables does not need to add an external factor to the system. Indeed, Frisch (2014) and Pearl (2009) suggest that the variable itself is the cause, and that the intervention amounts to changing the value of the variable. The interventionist accounts applies well to causal set theory in that, for example, tweaking the value of a variable corresponds to the addition of an element to the causal set—where the modal aspects of the interventionist account consists of the many possible ways in which one element can be added to a given set.

There are two possible issues with the interventionist account though. The first one is already pointed out by Ben-Menahem (2018) and maintains that the manipulation of a given variable to explain causal connections might be too limited at times. For example, many laws of physics that seem to convey causal connections are expressed in terms of hyperbolic differential equations. Even without entering the debate on what constitutes a physical law, we can simply consider the 1-d wave equation $u_{tt} - c^2 u_{xx} = 0$ which can be used to model a vast number of physical phenomena (plucked strings, vibrations of elastic beams, springs, and others). The mathematical form of those equations requires the input of some additional data (for example: position $u(x, 0) = f(x)$ and momentum $u_t(x, 0) = g(x)$) to obtain a well-defined solution. In general, these data are applied to the

dynamical equation to obtain a solution at a subsequent instant of time.

The interventionist approach seems to work fine with phenomena modeled by such equations, since the manipulation of the initial conditions determines a change in the solution of the equation at a subsequent instant of time. But, the account is also limited in that it is not always the case that the initial conditions required by the mathematical formalism are the sole cause for a given event. For example, the initial conditions of a projectile will determine when and where it will hit the target, but it is also easy to imagine how heavy rain and hail might affect its trajectory. Weather forecast is (typically) not accounted for by the initial conditions that describe the projectile, yet the latter can be considered as a co-cause for the projectile's final trajectory. This is to say that the initial conditions required by the mathematical formalism are necessary, yet not always sufficient to account for the causes of a physical system. Perhaps, one could add to the causal history of the system the extra causes (e.g., heavy rain and hail) that contribute to the corresponding effect. Then, to assess whether the new variables play the role of con-causes, one can verify that by changing the value of the variables corresponds a change in the phenomenon. In some cases though, the two variables X and Y—respectively the supposed cause and effect—could be related by association laws, namely laws that express mutual functional dependence between variables. In these cases, an intervention on X would produce a change in Y, but the other way around would also hold, thereby conflicting with the asymmetric character of causal relations.[10]

Another objection is raised in Blanchard (2023) and is based on the difficulty of addressing the causal efficacy of wholes and parts in the interventionist account. With respect to composite objects, the change of value of a

variable due to intervention ought to either keep the variables associated to the individual parts fixed, or to allow for a change of behavior of the individual parts. However, “[t]he first strategy runs the risk of making wholes causally excluded by their parts, whereas the second strategy is in danger of mistakenly ascribing to composite objects causal abilities that properly belong to their parts only” (Blanchard, 2023, 20).

The discussion on the interventionist account and modifications thereof is far from settled. Nonetheless, since it has been considered as a good fit for causal set theory, I contend that it falls short of giving a proper explanation of the causal relations. For example, consider the simplified causal set tree in Fig. 6.3. The variable subject to manipulation is the last event of the given node (causal set) from which the new event is born. Because of transitivity, the formation of a causal connection between the two events on the left-branch, that is the transition $(\bullet) \rightarrow (\vdots)$, can be considered as the cause for the top causal set $(\vdots\ \bullet)$. However, the same goes for the right-branch of the tree, that is, the transition $(\bullet) \rightarrow (\bullet\bullet)$ can be considered as the cause for the same top causal set. The interventionist account does not distinguish between the two branches as possible different causal transitions.[11] One could argue that both causal sets $(\vdots)$ and $(\bullet\bullet)$ (causally) contribute to the top one, but this would conflict with the fact that, while the different nodes are part of the kinematic scheme of the theory, they are not physical realizations.[12] The same argument runs against the counterfactual approach, for counterfactual statements would not be able to distinguish the cause of the realization of the top-node.

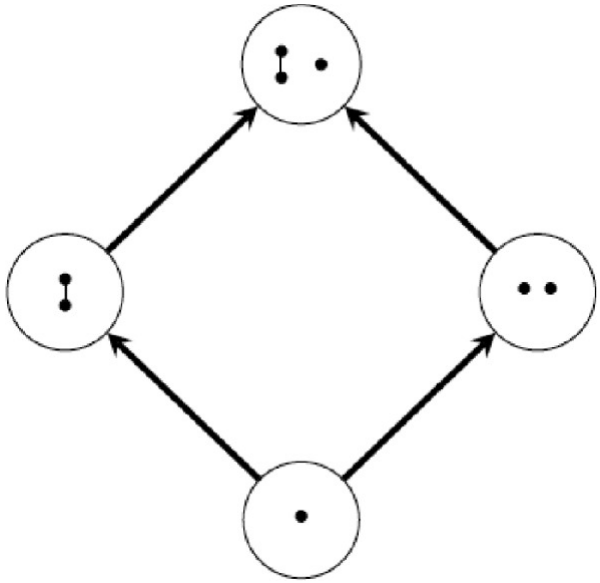


Fig. 6.3 A simplified causal set tree

Perhaps, to solve the impasse of characterizing causal relations, we can focus on the order relation among the events of the causal set. The binary relation $\prec$ is characterized as a partial order—this is evident also in other reviews of causal set theory such as Wüthrich (2012) and Dowker (2006). There, a causal set is defined by the pair $\langle E, \leq \rangle$ where E is the set of elements and $\leq$ is the binary relation of partial ordering between those elements. For example, Wüthrich (2012) defines the relation $\leq$ as inducing a partial order on a set $\mathcal{C}$, that is, the relation is transitive, reflexive and antisymmetric. Similarly, Dowker (2006) describes causal sets as partially ordered sets with a relation of precedence that satisfies: transitivity, non-circularity, and finiteness. Another example is Surya (2019), who defines a causal set as a set with an order relation $\prec$ that is acyclic, transitive, and locally finite.

Notably, Surya (2019, 12) defines the property of local finiteness as: $\forall x, y \in \mathcal{C}, |\mathbf{I}[x, y]| < \infty$, where $|\mathbf{I}[x, y]| \equiv Fut(x) \cap Past(y)$. The definition makes use of temporal parlance in the terms 'past' and 'future,' but one can easily avoid them by giving a definition that uses the notion of cardinality—as we mentioned in Sect. 6.2. Thus,

causation can be defined in terms of ordering between elements, and without making use of any spatiotemporal terms. This suggests that we should be able to give an account of the causal relation of causal set theory using an a-temporal approach to causation.

Most recently, Ben-Menahem (2018) defended a pluralist account of causation which has roots in the works by Cartwright (1983, 2004), Godfrey-Smith (2009) and others. Ben-Menahem (2018) characterizes causation in terms of constraints, similarly to how the lightcones of special relativity constrain the possible causal interactions for a given system. While causation "appears to be the basis for the very structure of spacetime" (Ben-Menahem, 2018, 28), the pluralistic account consists of approaching the notion of cause as a cluster of irreducible constraints imported from our theories. The view opposes the tradition of searching for a single notion of cause that can be applied to all sciences, and suggests that causal notions provide explanations and descriptions of possible changes in our physical world: "causal notions and constraints, I suggest, are employed to describe, predict, and explain change. They tell us which physical processes and changes in the physical world are possible, and which are not" (Ben-Menahem, 2018, 14).

We can thus interpret the partial order relation of causal set theory as a constraint over the possible growths in the kinematic space defined by the theory. This interpretation is compatible with the conclusions of the previous section—for which causal relations are more fundamental than spatiotemporal relations—and frees the concept of causation from temporal connotations. The a-temporal partial order between the elements of a given causal set is then embedded within the total kinematic space, and partakes to the dynamics that assigns a probability amplitude to each possible partially ordered growth. It remains that a quantum model for the interference

between the different nodes has not been developed yet, but the underlying intuition is to follow the analogy with the sum-over-histories account to suppress the un-physical causal sets (Carlip & Surya, 2023).

In sum, the interpretation of causal relations as constraints fits well with the three main claims of this contribution. The two levels of the theory, classical and quantum, are compatible with the constraint interpretation of the causal relation. Indeed, the elements of the individual sets are sorted by the partial order relation—whose definition can vary slightly depending on the formulation of the theory. The causal relation acts as a super-selection rule over all the possible growths by limiting those that would violate the conditions necessary to the recovery of relativistic metric. The second claim was that physical time in causal set theory is derived from the more fundamental causal relation. Again, this is also compatible with the constraint interpretation, for there is no addition of temporal terms and no changes in how physical time would emerge starting from causal sets. Finally, the constraint interpretation, under the pluralist view suggested by Ben-Menahem (2018), applies to the case of causal set theory, and it does not aim at becoming a general interpretation of causal relations across different disciplines. In addition, partial order relations are as clear as the spatiotemporal relations of relativity theory, thereby sidestepping the objection raised by Smart (1969).

6.5 Conclusion

In this contribution I have offered a brief overview of the main axioms of causal set theory, with a focus on the analogy with the sum-over-histories account, and on the relationship between the nodes representing individual causal sets and the causal tree representing the multiplicity of possible growths. I have then introduced the Malament-

Hawking theorem, which plays a central role in relating causal set theory to relativistic spacetime and its causal structure. After the review of the theory, I discussed the role of the time parameter in terms of bookkeeping device for the growth of new elements of a given set with respect to other events. What emerged is that it is possible to consider the causal relations as more fundamental than spacetime relation. Finally, after reviewing some accounts of causation in the context of philosophy of science, I suggested that starting from a pluralistic account of causation, we can interpret the relation of partial order as a constraint on the possible growths defined by the theory.

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Footnotes

1 In his version of causal set theory, Dribus (2017) replaces some of the axioms (e.g.: transitivity) and thus, for example, the top-left node in Fig. 6.1 is not part of the causal tree.

2 We shall specify the concept of growth later in this paper.

3 Forgione (2020) describes this mechanism of constructive and destructive interference of possible trajectories in the path integrals formulation of quantum mechanics.

4 Alternatively, a relativistic spacetime manifold that satisfies some appropriate causal conditions can be characterized by a causal relation $\leq_{GR}$ that is a strict partial order. While Malament (1977) identifies five such conditions, it is enough for our purposes to list the *past and future distinguishing condition* for which a "relativistic spacetime manifold X is future distinguishing if and only if for every x in X, and every open set U containing x,

there exists a ‘smaller’ open set U' in U, containing x, such that no future-directed smooth timelike curve through x that leaves U' ever returns to it” (Dribus, 2017, 101; Malament, 1977, 1400).

5 The use of the counting measure was already suggested by Myrheim (1978, 1): “If spacetime is assumed to be discrete, then the counting measure is the natural measure, and the causal counting is the only structure needed. Coordinates and metric may be derived as secondary, statistical concepts.”

6 This amounts to constructing a causal set from a pre-existing manifold, rather than obtaining the manifold bottom-up from the individual elements of the set.

7 In addition, the classical sequential growth dynamics imposes a locality condition called *Bell Causality*. The condition states that the probability of the growth of a new element depends only on its past, and it is not affected by spacelike separated elements. The condition guarantees a form of locality to the dynamics of the theory by imposing that the probability of growth of a new element does not depend on the totality of the causal set.

8 Notably, one might advance the objection for which mathematical objects might be in a spacelike relation with physical events, but a proper response would involve tackling the debate on the existence of mathematical entities, which I shall not do it here.

9 Notably, this is a difficulty proper of almost every theory of quantum gravity, and not only of causal set theory.

10 The argument was raised and commented in Kistler (2013).

11 That the two branches are physically indistinguishable is a consequence of gauge invariance. Nonetheless, this applies only to the physical content of the theory, and not to its modal aspects.

12 One could also reply that, as mathematical possibilities, every possible causal set contributes causally to the realization of a subsequent causal set. This would be similar to saying that the possible trajectories individuated by a path integral causally contribute to the total transition amplitude. This approach would require the backing of some form of mathematical realism, and we thus leave it to later works.

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7. Covariant Canonical Quantization and the Problem of Time

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Abstract
In the covariant canonical approach to classical physics, each point in phase space represents an entire classical trajectory. Initial data at a fixed time serve as coordinates for this “timeless” phase space, and time evolution can be viewed as a coordinate change. We argue for a similar view in quantum theory. As in the Heisenberg picture, the wave function is fundamentally time-independent. On any given time slice, however, we can diagonalize a complete set of position operators to form a basis, in which the projected wave function depends on the choice of time. In this picture, time evolution can be viewed as a basis change in

what is otherwise a block universe. We argue that this may help solve the “problem of time” in quantum gravity, and illustrate the idea with an example from three-dimensional quantum gravity.

7.1 Introducing the Problem of Time

The “problem of time” (Kuchař, 2011; Isham, 1993) is widely recognized as one of the fundamental conceptual obstacles to a quantum theory of gravity. Like other quantum theories, quantum gravity should presumably describe the time evolution of quantum states and operators. However, conventional approaches to quantum evolution fail rather dramatically. For a closed universe, for example, the standard time evolution operator, the Hamiltonian, is zero when acting on physical states, and the theory appears to be ‘frozen.’ For states that involve superpositions of sufficiently different physical configurations, it is not even clear that one can define a consistent “time” at all.

Time evolution is already subtle in classical general relativity. As a dynamical theory, general relativity describes the evolution of spacetime geometry from some initial spatial hypersurface (or “time slice”) to a later one. Nevertheless, the theory is generally covariant, with no fixed background structure, so our choice of time coordinate can be changed arbitrarily without affecting the physics. A time translation $t \to t + \Delta t$ in coordinate time can be viewed as nothing more than a coordinate change. Consistent with this picture, the Hamiltonian, which generates such transformations, is a constraint, vanishing on physical configurations.[1] This does not prevent us from describing time evolution in classical general relativity, of course. In practice, we do this by using some form of relational time. To describe the propagation of gravitational

waves, for instance, we assume a fixed background on which the waves are perturbations, and use that background to define a preferred time. To describe our Solar System, we start with an approximation in which the Sun defines a preferred reference frame, and treat planetary motion as small perturbations. To describe cosmology, we often use York's "extrinsic time" (Choquet-Bruhat & York, 1980), in which the trace of the extrinsic curvature—physically, local Hubble expansion rate—serves as a preferred time.

In quantum gravity, though, relational time becomes much more difficult. One can measure time with GPS satellite signals (Rovelli, 2002), for instance, or, reaching back in history, with the rotation of the Earth. But what if one's wave function contains configurations with no GPS satellites, or no Earth? One might postulate a cloud of ideal "clocks," as in Brown and Kuchař (1995), but these will back-react on the spacetime, changing the questions one is asking, and their quantum properties will make them imperfect time keepers (Unruh & Wald, 1989; Gambini et al., 2009).

We argue here that a partial solution can be found in a somewhat unconventional view of classical mechanics. The phase space of a classical theory, the starting point for canonical quantization, is usually described as the space of initial data on some time slice Σ_t, with additional structure (a symplectic form) determined by the details of the theory. In the covariant phase space picture (Ashtekar & Magnon, 1975; Ashtekar et al., 1991; Crnkovic & Witten, 1987; Crnkovic, 1988; Lee & Wald, 1990; Wald, 1993), in contrast, each point in phase space is an entire classical history, that is, a complete solution of the classical equations of motion. These two descriptions seem very different, but in a theory with a well posed initial value problem, they can be shown to be in one-to-one correspondence. Despite this correspondence, though, the

covariant phase space picture leads to a quantum theory that looks a bit unusual. Quantization automatically gives time-independent wave functions; in gravity, these are essentially amplitudes for entire block universes. But by classically mapping the covariant phase space to the conventional phase space on a time slice $\sum_t$, we can obtain a set of slice-dependent observables, which become "time"-dependent operators in the quantum theory. These operators, in turn, determine a "time"-dependent basis for the Hilbert space, different for each time slice $\sum_t$. The result is not unlike the usual Heisenberg picture, but without any need for a preferred time coordinate.

We will elaborate on this picture below, arguing in particular that the initial data on any time slice can serve as coordinates for the covariant phase space, with time evolution acting as a coordinate change. After quantization, such a coordinate change translates into a change of basis, where any classical choice of time slicing provides an admissible basis. As in conventional quantum mechanics, we can change from a Heisenberg picture to a Schrödinger picture by expressing wave functions in such a "time"-dependent basis. But we will now have an infinite number of Schrödinger pictures, one for each classical time slicing, all equivalent to the original Heisenberg picture. We will give a simple example from gravity in three spacetime dimensions in which this program can be carried out in full, and will conclude with a discussion of the possibilities and difficulties of extending the approach to realistic quantum gravity. We do not claim mathematical rigor—for many of our arguments, rigorous treatments are known only for simple cases—and we certainly do not pretend to have quantized gravity. Rather, our argument is that *if* general relativity can be quantized using covariant phase space methods, this might simplify some of problems associated with time in quantum gravity.

7.2 The Problem of Time in Quantum Gravity

We begin with a brief review of the problem(s) of time in quantum gravity. This is a large subject, and we refer readers to Kuchař (2011) and Isham (1993) for more comprehensive treatments and technical details. We will focus on the most straightforward approach to quantum gravity, Dirac quantization in the metric representation, but our conclusions carry over to other approaches, such as the connection representation of loop quantum gravity (Ashtekar & Bianchi, 2021).

Newtonian mechanics and nonrelativistic quantum mechanics assume an absolute time that, in principle, provides a reference to describe time evolution. Special relativity abandons absolute time, but still contains a collection of fixed, nondynamical reference frames that provide an unambiguous description of evolution. Even quantum theory in a curved spacetime has a fixed spacetime background to use as a reference for time evolution. In general relativity, on the other hand, spacetime is dynamical, with no preferred specification of time. The time coordinate t is just that: a human-made label with no intrinsic meaning, which can be chosen and changed arbitrarily. This arbitrariness shows up in several ways that we enumerate in the next subsections.

7.2.1 Hamiltonian Evolution

The Hamiltonian description of a physical system requires a choice of time coordinate, seemingly breaking general covariance. A long effort to address this problem culminated in the work of Arnowitt et al. (2008). In the ADM formalism, at time coordinate "t" is named, but is left completely unspecified except for the requirement that the

hypersurfaces of constant t be spacelike. A general spacetime metric can be written as[2]

$$ds^2 = N^2 dt^2 - q_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{7.1}$$

where q_{ij} is a positive definite spatial metric on each slice Σ_t of constant t. The Hamiltonian version of the Einstein–Hilbert action then takes the form

$$I = \int dt \int d^3x \left[\pi^{ij}\dot{q}_{ij} - N\mathscr{H} - N_i\mathscr{H}^i\right], \tag{7.2}$$

where π^{ij} is the momentum conjugate to q_{ij} and

$$\mathscr{H}^i = -2\,{}^{(3)}\nabla_j \pi^{ij}, \quad \mathscr{H} = \frac{2\kappa^2}{\sqrt{q}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) - \frac{1}{2\kappa^2}\sqrt{q}\,{}^{(3)}R. \tag{7.3}$$

Here $\kappa^2 = 8\pi G$ is a constant, ${}^{(3)}\nabla$ is the spatial covariant derivative compatible with q_{ij}, $\pi = \pi^i_i$, and ${}^{(3)}R$ is the intrinsic scalar curvature of the slice Σ_t. The action (7.2) is a standard canonical action, and the Hamiltonian

$$H = \int d^3x (N\mathscr{H} + N_i\mathscr{H}^i) \tag{7.4}$$

gives the correct Hamilton's equations of motion. But the 'lapse function' N and the 'shift vector' N_i appear only as Lagrange multipliers, whose equations of motion are $H = 0$. Up to possible boundary terms, the Hamiltonian for general relativity is thus a *constraint*, vanishing for physical configurations. This reflects the fact that the time coordinate t is arbitrary; a translation in t may look like time evolution, but it can be implemented by merely changing coordinates. In classical general relativity, we have learned to live with this problem. In a setting in which we have a time coordinate with a clear physical meaning—for instance, in most cosmological applications—we can think of the coordinate t as being a physical clock time, with evolution defined relative to this clock. In quantum

gravity, though, the situation is more complex. The usual procedure for obtaining a Schrödinger equation no longer gives us a time evolution equation, but rather the Wheeler–DeWitt equation (DeWitt, 1967)

$$\widehat{H}|\psi\rangle = 0. \tag{7.5}$$

The time derivative in the usual Schrödinger picture has disappeared, leaving us with a "frozen" formalism in which the physical states appear to be independent of time. We will see below that the same frozen time also occurs in the Heisenberg picture. Mimicking the classical procedure, we can attempt to extract a relational time from (7.5), but efforts in this direction have met with little success. While there are natural choices of time for particular spacetimes, a wave function should allow a superposition of many geometries, and it is not at all clear how to extract a single relational time that makes sense for an arbitrary superposition. For many years, it was widely hoped that York's "extrinsic time" (Choquet-Bruhat & York, 1980), in which the mean extrinsic curvature K serves as a time coordinate, might give a universal slicing. But we have now learned that an infinite family of spacetimes admit no such hypersurfaces (Chrusciel et al., 2005). Other attempts have fared equally poorly, making sense for only limited classes of spacetimes or failing to pick out unique constant time hypersurfaces.

7.2.2 The Inner Product

In addition to an evolution equation, a quantum theory must provide an inner product on the space of states, in order to define probability amplitudes and normalize probabilities. Here, too, the absence of a fixed background time causes trouble.

Let us suppose we have managed to solve the Wheeler–DeWitt equation (7.5), obtaining a space $\{|\psi_\alpha[q]\rangle\}$ of

physical states, which for simplicity we will take to be functions of the spatial metric q_{ij} (the metric representation). The metric q_{ij} is defined in (7.1) at a particular time t, that is, on a particular time slice Σ_t. Intuitively, we would like the inner product $\langle\psi_\alpha|\psi_\beta\rangle$ to refer to wave functions at "the same time." But the time coordinate t and the slice Σ_t have completely dropped out of the Wheeler–DeWitt equation, and there is no background spacetime to tell us what "the same time" means. This problem manifests itself mathematically in the fact that the naive inner product

$$\langle\psi_\alpha|\psi_\beta\rangle = \int [dq]\psi^*_\alpha[q]\psi_\beta[q]$$

is badly divergent. The divergence comes from the fact that infinitely many spatial metrics q_{ij} represent the same physical configuration, defined on time slices that differ from each other only by coordinate changes. The Hamiltonian constraint $\hat{H}$ generates "surface deformations" that move the slice Σ_t, dragging along the metric, without changing the actual physics (Isham & Kuchař, 1985), and the naive inner product counts all of them. In principle, we understand how to deal with this problem: we should gauge fix the inner product, essentially restricting the metric q_{ij} to a single time slice (Woodard, 1993). In practice, though, this requires finding a universal time slicing, a "clock" that makes physical sense for any spacetime that can appear in a superposition of states. Once again, we do not know how to do this.

7.2.3 Operators and Causal Structure

The third ingredient needed for a quantum theory is a set of observables, self-adjoint operators that take physical states to physical states. Even in this case a problem of time appears. Suppose $|\psi\rangle$ is a physical state, that is, a

state obeying (7.5), and let $\widehat{\mathcal{O}}$ be an operator. For $\widehat{\mathcal{O}}$ to take physical states to physical states, we must have $\widehat{H}\widehat{\mathcal{O}}|\psi\rangle = 0$, which implies in turn that

$$\left[\,\widehat{H}, \widehat{\mathcal{O}}\,\right] = 0 \quad \text{on physical states.} \tag{7.6}$$

Such operators exist and it may be possible to define them by projection or related constructions (Chataignier, 2020, 2023; Goeller et al., 2022), but they are necessarily nonlocal (Torre, 1993; Donnelly & Giddings, 2016), thus making their physical interpretation problematic (Tsamis & Woodard, 1992). This is not merely a technical issue; it goes to the heart of the problem of causal structure in quantum gravity. In standard quantum field theory, the causal structure of spacetime is enforced by the condition of "microcausality":

$$\left[\,\widehat{\mathcal{O}}_1(x), \widehat{\mathcal{O}}_2(x')\,\right] = 0 \quad \text{if spacetime points } x \text{ and } x' \text{ are spacelike separated.} \tag{7.7}$$

Such a condition is problematic in a quantum theory of gravity: whether x and x' are spacelike separated or not depends on the metric, which is no longer fixed. The absence of local operators is an explicit manifestation of this problem. As noted above, (7.6) also implies that the Heisenberg picture for quantum gravity, just like the Schrödinger picture, is "frozen." Indeed, the Heisenberg equation of motion for a physical operator is now

$$\frac{d\widehat{\mathcal{O}}}{dt} = \frac{i}{\hbar}\left[\,\widehat{H}, \widehat{\mathcal{O}}\,\right] = 0. \tag{7.8}$$

7.3 Covariant Phase Space

We now turn to a rather different topic, although one that will ultimately link up with the problem of time. The covariant phase space description of a classical system can be traced back to Lagrange (see Ashtekar et al., 1991), but

has recently gained traction as a useful way of obtaining a canonical description while preserving manifest general covariance. We again give a brief review, referring readers to Ashtekar and Magnon (1975), Ashtekar et al., (1991), Crnkovic and Witten (1987), Crnkovic (1988), Lee and Wald (1990), Wald (1993), Hu (2022) and Carlip (2020) for more comprehensive treatments and technical details.

The standard setting for Hamiltonian mechanics is phase space $\mathcal{M}$, conventionally defined as the space of initial data on a d-spatial-dimensional Cauchy surface Σ_t. But for a theory with a well posed initial value problem, each set of initial data determines a unique classical solution, a classical "history." The basic idea of covariant phase space is to reformulate phase space as the space of such histories.

More precisely, in the standard description initial data can be parametrized by canonically conjugate generalized positions and momenta (φ^i, π_i), which can together be viewed as coordinates z^A on $\mathcal{M}$. The space $\mathcal{M}$ comes equipped with a symplectic current ω, a closed d-form (that is, a rank d differential form for which $d\omega = 0$),

$$\omega = \delta\pi_i \wedge \delta\varphi^i = \omega_{AB}\delta z^A \wedge \delta z^B, \tag{7.9}$$

where variations δz of fields are treated as exterior derivatives on phase space $\mathcal{M}$. Such a form can be integrated over a d-dimensional surface; its integral over a Cauchy surface Σ_t,

$$\Omega = \int_{\Sigma_t} \omega,$$

is the symplectic form on $\mathcal{M}$, a two-form on phase space.[3] Up to subtleties concerning boundary terms, the fact that ω is closed ensures that Ω is independent of the choice of Cauchy surface Σ_t.

For a theory with gauge symmetries, Ω is degenerate—gauge variations $\delta_\lambda z$ are null directions—and Ω is technically a "presymplectic form." This requires a bit more work to quotient out the degeneracies (Lee & Wald, 1990), a process known as symplectic reduction. We will not need this level of detail here, but there is one subtlety that may be relevant to quantum gravity. In order for symplectic reduction to work, the gauge symmetry must act "nicely" on the phase space, so the quotient space is typically a stratified manifold. For large classes of solutions of the vacuum Einstein equations, this is guaranteed by slice theorems (Ebin, 1970; Isenberg & Marsden, 1982). But there are known truncations of general relativity with certain matter couplings in which these theorems fail (Dittrich et al., 2017). In such situations, the covariant phase space construction described here may also fail.

Given a symplectic form Ω, we can now write down Poisson brackets between functions on phase space,

$$\{X, Y\} = \int_{\Sigma_t} \frac{\delta X}{\delta z^A} (\omega^{-1})^{AB} \frac{\delta Y}{\delta z^B}. \tag{7.10}$$

If we are given a family of transformations—time translations, for instance—of the form $\delta_\tau z^A$, labeled by a parameter τ, Hamilton's equations of motion become

$$\delta_\tau z^A = (\omega^{-1})^{AB} \frac{\delta H[\tau]}{\delta z^B}, \tag{7.11}$$

which can be taken as a definition of the Hamiltonian for the transformation. Slightly more obscurely, (7.11) can be written as

$$\delta H[\tau] = \Omega[\delta z, \delta_\tau z], \tag{7.12}$$

the defining equation for the Hamiltonian. Suppose now that we are dealing with a system with a well posed initial value problem, that is, a system for which any set of initial data (φ^i, π_i) determines a unique solution. Then each point

in the phase space $\mathcal{M}$ defined on a Cauchy surface Σ_t determines a classical solution Φ. Conversely, given a Cauchy surface Σ_t, we can restrict any classical solution to that surface to determine a point in $\mathcal{M}$. This provides a one-to-one map between $\mathcal{M}$ and the space $\overline{\mathcal{M}}$ of classical solutions, or histories. Indeed, consider the map

$$\tau_{\Sigma_t} : \overline{\mathcal{M}} \to \mathcal{M} \qquad \Phi \mapsto (\varphi^i, \pi_i)\,|_{\Sigma_t} \tag{7.13}$$

that takes any classical solution to its initial data on the slice Σ_t. The well-posedness of the initial value problem tells us that this map is invertible, and requiring it to be a diffeomorphism determines a topology on $\overline{\mathcal{M}}$ (although with some mathematical subtleties; see, for instance, Woodhouse, 1992, Chap. 2).

The space of classical solutions has a natural symplectic form $\overline{\Omega}$, defined entirely from the variation of the spacetime action (Crnkovic & Witten, 1987; Crnkovic, 1988; Lee & Wald, 1990; Wald, 1993). We will not need details here; see Appendix A of Carlip (2020) for a short review. Moreover, the map (7.13) is a symplectomorphism, taking $\overline{\Omega}$ to Ω and thus preserving the phase space structure. The pair $(\overline{\mathcal{M}}, \overline{\Omega})$ is commonly known as "covariant phase space," covariant in the sense that it requires no choice of a time slice Σ_t.

For a large class of diffeomorphism invariant theories, including general relativity, one can find an explicit expression for a Hamiltonian $\overline{H}$, obeying (7.12), that generates diffeomorphisms (Lee & Wald, 1990; Wald, 1993). This $\overline{H}$ is the covariant phase space version of the ADM Hamiltonian (7.4), and is again a constraint, vanishing (up to possible boundary terms) on physical solutions. Note that while $(\overline{\mathcal{M}}, \overline{\Omega})$ and $(\mathcal{M}, \Omega)$ are isomorphic, they are not *canonically* isomorphic. The isomorphism depends on the choice of Cauchy surface Σ_t, and in general there is no preferred choice. This has two implications (Hu 2022; Woodhouse, 1992):

1. Points in the conventional phase space $\mathcal{M}$ can be thought of as coordinates for $\overline{\mathcal{M}}$. While it is sometimes possible to find a global parametrization of classical solutions, for instance in terms of constants of motion, this is often difficult. But initial data on a fixed time slice provide a useful set of labels, and we can interpret τ_{Σ_t} as a coordinate map.

2. Consider a family $\Sigma_{\{t\}}$ of Cauchy surfaces. Suppose we are given initial data (φ^i, π_i) on Σ_t and want to the fields at a later slice $\Sigma_{t'}$. From (7.13),

$$(\varphi^i, \pi_i)\,|_{\Sigma_{t'}} = \tau_{\Sigma_{t'}} \circ \tau_{\Sigma_t}^{-1}(\varphi^i, \pi_i)\,|_{\Sigma_t} \tag{7.14}$$

   But this is simply a change of coordinates on $\overline{\mathcal{M}}$. We can thus view time evolution as a coordinate change on covariant phase space.

   Note that we have made no assumptions about Σ_t and $\Sigma_{t'}$ except that they are Cauchy surfaces. This does not, of course, absolve us from solving any equations—the covariant phase space approach implicitly assumes that we know the full classical solutions. But it does provide us with a Hamiltonian description of evolution that requires no preferred choice of time slicing.

We can thus treat phase space either as a space of initial data or as a space of classical histories. Mathematically the two descriptions are isomorphic. Physically, however, they give very different pictures. The ordinary phase space picture is one of evolution in time: one starts with initial data at a fixed time and determines its future development. The covariant phase space picture is inherently time independent: its elements are entire histories, and "time" is merely a coordinate choice. There is nothing in the physics that prefers one view over the other. But the covariant phase space formalism is perhaps a bit broader in its reach,

requiring a space of solutions but not necessarily a good initial value problem.

7.4 Covariant Canonical Quantization

The question is now whether this classical picture of covariant phase space can be used to build a quantum theory. For simple enough cases—for instance, free scalar fields in curved spacetime (Ashtekar & Magnon, 1975), (2+1)-dimensional gravity with simple topologies (Carlip, 1990, 1992, 1993; Carlip & Nelson, 1995, 1999; Kraus et al., 2021), some models of two-dimensional gravity (Navarro-Salas et al., 1993), and certain truncations of general relativity (Hu, 2022)—this is known to be possible. For more complicated theories, such as the full general theory of relativity, it is less clear. For instance, without a general solution of the field equations we cannot fully characterize the covariant phase space, although there may be sensible ways to use classical perturbation theory (Basu, 2005).

We will argue here that if such a quantization is possible, it will provide a major step towards solving the difficulties described in Sect. 7.2. While the problem of time would not be entirely eliminated, it would be reduced to the more tractable problem of time in classical general relativity. Let us suppose we have found, at least locally, a canonical set of coordinates $\{\bar{\varphi}^a, \bar{\pi}_a\}$ on covariant phase space $\overline{\mathcal{M}}$, that is, a set for which the symplectic form is $\overline{\Omega} = \delta\bar{\pi}_a \wedge \delta\bar{\varphi}^a$. Typically, $\overline{\mathcal{M}}$ is infinite dimensional; the index a corresponds roughly to the pair (i, x) in the standard canonical formulation. For simplicity, we are taking this index to be discrete—for instance, labeling coefficients of a basis of functions—but a generalization should be possible. We will assume the simplest form of

quantization, in which the Poisson brackets of these coordinates become commutators,

$$\begin{aligned}\left[\hat{\bar{\varphi}}^a, \hat{\bar{\pi}}_b\right] &= i\hbar\,\delta^a_b,\\ \left[\hat{\bar{\varphi}}^a, \hat{\bar{\varphi}}^b\right] &= \left[\hat{\bar{\pi}}_a, \hat{\bar{\pi}}_b\right] = 0,\end{aligned} \tag{7.15}$$

and we will take our wave functions $|\Psi[\bar{\varphi}]\rangle$ to be in the position representation, that is, functions of the generalized positions $\bar{\varphi}$. There are, of course, other possibilities—one might choose different fundamental variables and commutators, for instance Isham (1984), or use a more sophisticated technique such as deformation quantization (Twareque & Englis, 2005)—but these are universal issues in quantization, not particular to covariant phase space. As in Sect. 7.2, we must also provide an inner product on the space $\{|\Psi[\bar{\varphi}]\rangle\}$ of wave functions, which will normally require gauge fixing the naive inner product. Some ideas of how to do this have been proposed (Woodard, 1993; Chataignier, 2020, 2023; Giulini & Marolf, 1999), but for quantum gravity the problem remains very difficult; we will assume it has been solved, at least in some systematic method of approximation.

If our theory is diffeomorphism invariant, the classical Hamiltonian $\overline{H}$ of Sect. 7.3 will still be a constraint, and instead of a Schrödinger equation, a Wheeler–DeWitt equation

$$\widehat{\overline{H}}\,|\Psi[\bar{\varphi}]\rangle = 0 \tag{7.16}$$

will again hold. Now, however, this condition has a simple interpretation: as functions on the space of histories, wave functions determine amplitudes for entire block universes, and are thus inherently time independent. The question is whether we can now obtain a picture that exhibits time evolution. To do so, start with the classical description of $\overline{\mathcal{M}}$, and pick a Cauchy surface Σ_t. Up to ordering

ambiguities, the map τ_{Σ_t} of (7.13) translates to a map between operators in the quantum theory,

$$\hat{\tau}_{\Sigma_t} : (\hat{\bar{\varphi}}^a, \hat{\bar{\pi}}_a) \mapsto (\hat{\varphi}^i(\Sigma_t), \hat{\pi}_i(\Sigma_t)). \tag{7.17}$$

This gives us a set of slice-dependent operators $\{\hat{\varphi}^i(\Sigma_t), \hat{\pi}_i(\Sigma_t)\}$, which depend on the covariant phase space operators $\{\hat{\bar{\varphi}}^a, \hat{\bar{\pi}}_a\}$ and the *classical* data that determine the map τ_{Σ_t}. These are examples of what Rovelli has called "evolving constants of motion" (Rovelli, 1991), operators that commute with the generator of time translations but depend on a "time" parameter—here, the slice Σ_t—that is determined by the classical dynamics. From the canonical commutation relations (7.15) and the fact that τ_{Σ_t} is a symplectomorphism, it follows that the operators $\hat{\varphi}^i(\Sigma_t)$ commute with each other, and should at least formally comprise a maximal set of commuting operators. Diagonalizing them thus gives us a new "evolving basis,"

$$\hat{\varphi}^i(\Sigma_t)|\varphi^i(\Sigma_t)\rangle = \varphi^i(\Sigma_t)|\varphi^i(\Sigma_t)\rangle, \tag{7.18}$$

which depends on the classical choice of Σ_t. Any covariant canonical state $|\Psi[\bar{\varphi}]\rangle$ can be expanded in this basis, and the components

$$\langle\varphi^i(\Sigma_t)|\Psi[\bar{\varphi}]\rangle$$

are "time" dependent, depending on the classical slice Σ_t. Moreover, if we choose a family $\Sigma_{\{t\}}$ of slices that foliate some region of classical spacetime, completeness of the bases $|\varphi^i(\Sigma_t)\rangle$ and $|\varphi^i(\Sigma_{t'})\rangle$ implies that the functions $\langle\varphi^i(\Sigma_t)|\Psi[\bar{\varphi}]\rangle$ and $\langle\varphi^i(\Sigma_{t'})|\Psi[\bar{\varphi}]\rangle$ are related by a formally unitary transformation. For any particular choice of time slicing, we can therefore write

$$i\hbar\frac{d}{dt}\langle\varphi^i(\Sigma_t)|\Psi[\bar{\varphi}]\rangle = \widehat{\mathcal{H}}(\Sigma_{\{t\}})\langle\varphi^i(\Sigma_t)|\Psi[\bar{\varphi}]\rangle \tag{7.19}$$

for some Hermitian operator $\widehat{\mathcal{H}}(\Sigma_{\{t\}})$. Note that $\widehat{\mathcal{H}}$ is *not* the Hamiltonian constraint of (7.16), and that it will generally differ for different choices of the family $\Sigma_{\{t\}}$. For any such family, the $\widehat{\mathcal{H}}(\Sigma_{\{t\}})$ are again evolving constants of motion, defined on the covariant phase space but depending on a classical choice of foliation.

Now consider an initial slice Σ_i and a final slice Σ_f. There are infinitely many ways to foliate the spacetime between them, each with its own $\widehat{\mathcal{H}}(\Sigma_{\{t\}})$. This proliferation of Hamiltonians leads to the "multiple choice problem" of time (Kuchař, 2011; Isham, 1993): one must worry whether the ultimate evolution from Σ_i to Σ_f depends on one's intermediate choices. Here, though, at least in principle this is not a problem: by construction, the final wave function $\langle\varphi^i(\Sigma_f)|\Psi[\bar{\varphi}]\rangle$ depends only on the covariant phase space state $|\Psi[\bar{\varphi}]\rangle$ and the final slice.

As we cautioned in the introduction to this Chapter, these argument are not mathematically rigorous. For simple examples like those of Hu (2022), the constructions can be carried out in full. But for more realistic cases, many technical problems remain. We have assumed that an inner product on the covariant Hilbert space has been found; for gravity this is an unsolved problem. The passage (7.17) from functions to operators is plagued by operator ordering ambiguities. Arguments about completeness of bases and unitarity become much more delicate for operators with continuous spectra. Results that are clear for finite dimensional Hilbert spaces do not always extend easily; even in a system as simple as a free scalar field in more than two dimensions, infinite sums involved in basis changes can reintroduce a multiple choice problem (Torre & Varadarajan, 1999). For quantum gravity, a first attempt at a rigorous analysis was made in Cosgrove (1996), but work since then has largely focused on simpler models. One

might hope, though, that the remaining problems, while difficult, are in some sense “technical,” and that the general conceptual approach to the problem of time will survive and more detailed mathematical analysis.

7.5 An Example from (2+1)-Dimensional Gravity

We now turn to a specific application of this formalism, quantum gravity for a spatially compact universe in 2+1 dimensions, that is, two dimensions of space plus one of time. It has been understood for 50 years (Staruszkiewicz, 1963) that such a reduction in dimension vastly simplified general relativity (see Carlip, 2005 for a review). In 2+1 dimensions the vacuum Einstein equations imply that spacetime is flat—or constant curvature if the cosmological constant is nonzero—and thus contains no propagating degrees of freedom.[4] But while this drastically simplifies the theory, it does not make it completely trivial: a topologically nontrivial spacetime will still carry global geometric degrees of freedom, which introduce the same conceptual problems we face in realistic (3+1)-dimensional gravity.

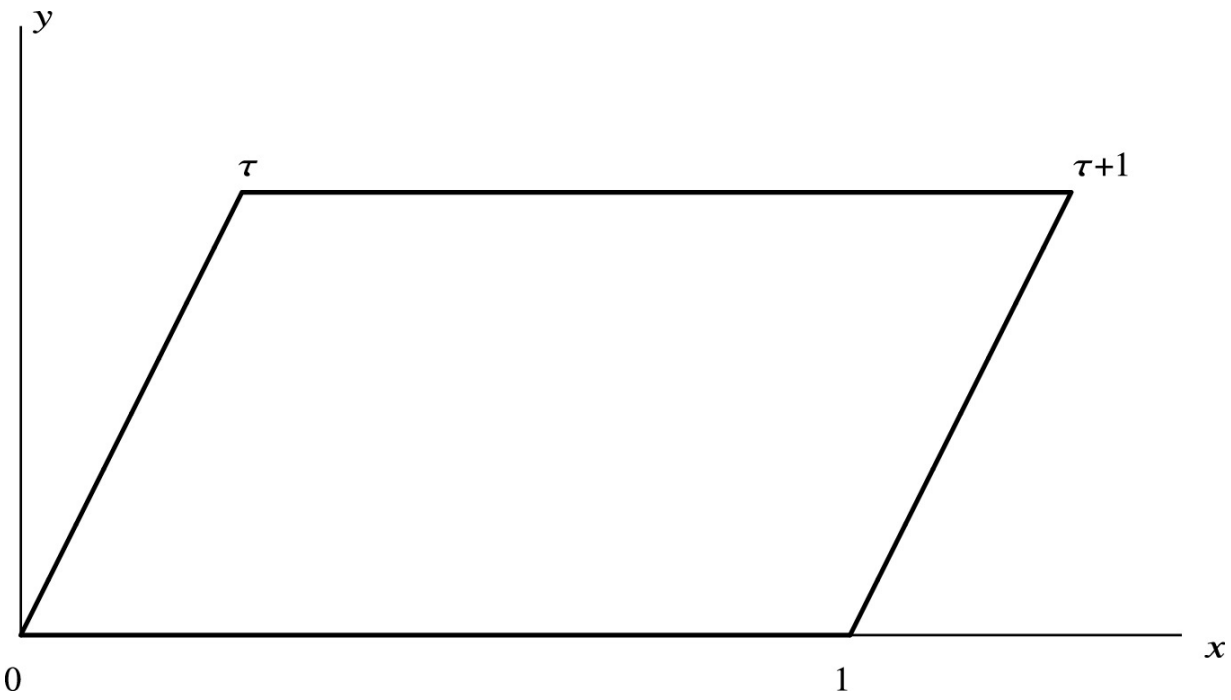

Fig. 7.1 A flat torus may be formed by "gluing" opposite edges. The complex modulus τ determines the global geometry; in general, tori with different moduli are not diffeomorphic

As a warm-up, consider a flat two-dimensional torus T^2. Such a space can always be constructed by cutting out a parallelogram from the plane and identifying opposite edges. After rescaling one circumference to 1, distinct tori are parametrized by a single complex number τ, the modulus, as shown in Fig. 7.1. In the coordinates shown in the figure, the metric is simply

$$ds^2 = |dx + \tau dy|^2 , \tag{7.20}$$

with edges identified by translations $x \sim x + 1$ and $y \sim y + 1$. These identifications are isometries of the Euclidean plane, as they must be to maintain flatness. The metric (7.20) is manifestly flat, but in general there is no diffeomorphism that will take a metric with one modulus τ to a metric with another: the modulus is a genuine geometric degree of freedom.

Now consider a flat (2+1)-dimensional spacetime with topology $\mathbb{R} \times T^2$, that is, a spatially closed three-

dimensional torus universe. Just as a two-dimensional torus can be obtained from a region of the plane by isometrically identifying the edges, any flat $\mathbb{R} \times T^2$ can be obtained from a region of flat (2+1)-dimensional Minkowski space by isometrically identifying the edges. It can be shown that the allowable isometries take the form (Carlip, 1990, 1992, 1993; Carlip & Nelson, 1995, 1999)

$$\begin{aligned} \Lambda_1 &: (t, x, y) \to (t \cosh \lambda + x \sinh \lambda, x \cosh \lambda + t \sinh \lambda, y + a) \\ \Lambda_2 &: (t, x, y) \to (t \cosh \mu + x \sinh \mu, x \cosh \mu + t \sinh \mu, y + b), \end{aligned} \tag{7.21}$$

now labeled by four real parameters (μ, λ, a, b). These parameters act as coordinates for the covariant phase space, uniquely labeling classical solutions with the specified topology. Moreover, they come with a natural symplectic structure $\overline{\Omega}$, obtained as in Sect. 7.3 and mathematically related to the symplectic structure of loops on a surface (Goldman, 1988). The quantization described in Sect. 7.4 is then straightforward: the nonvanishing commutators are

$$[\hat{\mu}, \hat{a}] = [\hat{b}, \hat{\lambda}] = -\frac{i\hbar}{2}, \tag{7.22}$$

and we can thus take our wave functions to be functions $\Psi(\lambda, \mu)$, with

$$\hat{a} = \frac{i\hbar}{2} \frac{\partial}{\partial \mu}, \qquad \hat{b} = -\frac{i\hbar}{2} \frac{\partial}{\partial \lambda}. \tag{7.23}$$

The wave functions $\Psi(\lambda, \mu)$ are time independent, and give amplitudes for entire block universes. To obtain time-dependent operators, we follow the procedure of Sect. 7.4, starting with a choice of classical time slicing. Here there is a natural choice; in this (2+1)-dimensional setting, York's extrinsic time always exists, and gives a unique slicing $K = -t$ (Andersson et al., 1997). Moreover, the slices of constant time t are spatially flat. With this choice, space at a fixed time is thus a flat torus, characterized by a modulus

$\tau(t)$ and its conjugate momentum $p(t)$. By looking at the explicit form of the spacetime metric, we can obtain this quantities in terms of the parameters (μ, λ, a, b), both classically and quantum mechanically:

$$\hat{\tau}(t) = \left(\hat{a} + \frac{i\hat{\lambda}}{t}\right)^{-1} \left(\hat{b} + \frac{i\hat{\mu}}{t}\right), \quad \hat{p}(t) = -it\left(\hat{a} - \frac{i\hat{\lambda}}{t}\right)^{2}. \quad (7.24)$$

As in Sect. 7.4, we can now diagonalize the operators $\hat{\tau}$ and $\hat{\tau}^{\dagger}$, finding eigenstates for which $\hat{\tau}|\psi\rangle = \tau|\psi\rangle$ and $\hat{\tau}^{\dagger}|\psi\rangle = \bar{\tau}|\psi\rangle$. A straightforward calculation (Carlip, 1990, 1992, 1993; Carlip & Nelson, 1995, 1999) yields

$$\langle\lambda, \mu|\tau, \bar{\tau}, t\rangle = K(\lambda, \mu|\tau, \bar{\tau}, t) = \left(\frac{\mu - \tau\lambda}{\tau_2^{1/2} t}\right) \exp\left\{\frac{-i|\mu - \tau\lambda|^2}{\tau_2 t}\right\}. \quad (7.25)$$

The slice-dependent wave functions $\Psi(\tau, \bar{\tau}, t) = \int d\lambda d\mu\, K(\lambda, \mu|\tau, \bar{\tau}, t)\Psi(\lambda, \mu)$ can then be showed to satisfy a Schrödinger equation

$$i\frac{d}{dt}\Psi(\tau, \bar{\tau}, t) = \widehat{\mathcal{H}}\Psi(\tau, \bar{\tau}, t) \quad \text{with} \quad \widehat{\mathcal{H}} = t^{-1}\left(\Delta_{1/2}\right)^{1/2}, \quad (7.26)$$

where $\Delta_{1/2}$ is a Laplacian on moduli space. This essentially reproduces the result one would obtain by first fixing the York time slicing and then quantizing (Carlip, 1990). We say "essentially" because the operator ordering in (7.24) is not unique, and different choices lead to $\mathcal{O}(\hbar)$ modifications of the Hamiltonian. For (2+1)-dimensional gravity, though, an additional symmetry reduces the ambiguity. The mapping class group—the group of "large" diffeomorphisms, diffeomorphisms that cannot be continuously deformed to the identity—acts nontrivially on the space of moduli τ, and the wave function should transform under a representation of this group. It turns out (Carlip, 1992, 1992; Carlip & Nelson, 1999) that the choice of representation completely determines the operator

ordering, the inner product, the normalization of the wave functions (7.25), and the Hamiltonian, all of the ingredients needed for a sensible quantum theory.

We thus have an explicit example in which a covariant phase space quantum theory of gravity, with no intrinsic time dependence, can be transformed into a “time”-dependent Schrödinger picture. This Schrödinger gives a physically sensible quantum theory: for instance, the wave functions (7.25) are peaked on the classical trajectories (Ezawa, 1994). But we have found the time dependence using only classical properties, in a way that would clearly work as well for any other classical choice of time slicing.

7.6 Conclusions

Covariant canonical quantization does not solve the problem of time in quantum gravity. It does, however, recast some of the questions in a more tractable form. In particular, one need not pick a preferred time slicing, classically or quantum mechanically. Rather, at least in principle, any classical choice of time slicing provides a set of slice-dependent operators that determine a Schrödinger picture time evolution for that foliation. The approach also offers a physicists’ perspective on the “block universe/evolving universe” question, although perhaps not one that addresses all of the philosophical issues. In covariant canonical quantization, there are an infinite number of descriptions of an evolving universe, one for each time slicing. But these are all isomorphic to a timeless block universe description coming from the quantization of the covariant phase space, and one’s choice of which description to use seems to have no physical content. Classically, this isomorphism requires that the systems under consideration be deterministic—that is the significance of a well posed initial value problem—but it is hard to see how to do physics without such an assumption.

Quantum mechanically the equivalence is much less clear in other formulations, but here it is automatic.

All this comes with a proviso: "assuming the program can be carried out." We do not wish to understate the potential problems. To fully define the covariant phase space, we need a complete characterization of the classical solutions, although we may be able to make progress with classical perturbation theory (Basu, 2005). In the example of Sect. 7.5 we had a diffeomorphism-invariant parametrization of the space of solutions, but if we do not—if, for instance, we use initial data as coordinates as in Sect. 7.3—we will need to gauge fix the inner product (Woodard, 1993), a process easy to describe but difficult to implement. To obtain a time-dependent picture we need a classical time slicing; an infinite number of such slicings exist, but we do not know to explicitly describe even one in a way that holds for the entire space of solutions. Physical operators on a slice will be nonlocal (as the moduli τ of Sect. 7.5 were), and hard to construct. And as in Sect. 7.5, we expect serious operator ordering ambiguities in the Hamiltonian, though perhaps these will be limited by the demand for consistent evolution (Cosgrove, 1996).

If we are lucky, though, these problems are "technical," requiring hard work but no radical new ideas. One could imagine, for instance, a systematic classical perturbative expansion of the field equations—the post-Minkowskian expansion, for instance of Damour (1987)—combined with an order by order specification of an inner product and spatial slicing. Whether this is possible remains to be seen. But if it is, then perhaps some of the conceptual problems of quantum gravity are not quite as severe as we have believed.

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Footnotes

1 In a spacetime with an asymptotic boundary, this is no longer quite true: the Hamiltonian acquires boundary terms, allowing a more traditional picture of time evolution, though only at the boundary. This does not seem relevant for our universe, however, which appears to be asymptotically de Sitter, with no boundary.

2 We use the conventions of Carlip (2019), which assume a metric with signature $(+---)$, a choice that affects some signs.

3 Beware of a potential source of confusion: ω is both a *d*-form on Σ_t *and* a two-form on the infinite-dimensional phase space $\mathcal{M}$.

4 A (2+1)-dimensional spacetime with an asymptotic boundary may have "boundary gravitons" that propagate by the boundary Hamiltonian, but as noted earlier, we are not dealing with that case here.

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8. Asymptotic Reasoning and Universality in (Space)Time Dynamics

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Abstract
The theory of general relativity (GR) displays an array of universality features that are central to the operational use of the theory, such as detecting and extracting information from gravitational waves, while at the same time connecting to foundational issues, e.g., in the gravitational collapse setting or in the high-frequency limit of the theory

(in the geometric optics approximation). Quite remarkably, these features can smooth out the details at a certain scale, namely through dissipative/smoothing mechanisms, in a way ensuring some level of predictive power (and simplicity) for the theory even when accurate control of the details is not at reach, either in principle or for operational reasons. We illustrate the role of such universality and dissipative/smoothing mechanisms in a series of examples, with a focus on binary black hole mergers, and we argue for the fruitfulness of (various forms of) asymptotic reasoning for identifying and investigating universal patterns in the GR context. In a broader perspective, we suggest that these latter may provide fundamental insights into the dynamics of (space)time in GR and beyond, in a way that is 'agnostic' to the underlying fundamental (e.g., quantum gravity) ontology.

8.1 Time and Dynamics: An 'Asymptotic Reasoning' Approach

The inquiry about the nature of *time* and, more specifically, its relationship with *dynamics*, understood as the systematic characterisation of the very notion of 'evolution' of a given 'physical system', has driven a substantial part of the development of physics. Our understanding of time has accordingly conformed to the prevailing theory of dynamics. In this sense, conceptions of time as either 'fundamental' or 'emergent' in physical theories crucially depend on how the notion of evolution is implemented in the respective theories.

In this contribution, we show that certain 'operational'—epistemic—features of the spacetime dynamics of the theory of general relativity (GR) can actually provide relevant insights into time, irrespective of whether time is conceived as fundamental or emergent at some deeper

level (e.g., quantum gravity or any other alternative fundamental approach to the gravitational phenomenon). To this aim, we will adopt an approach akin to what has been called ‘asymptotic reasoning’ in the literature (Batterman, 1997, 2002), where the study of dynamics makes use of asymptotic methods to unveil (in a precise sense) certain universal patterns that may be part of an underlying, more fundamental description. In Sect. 8.2, we will make a ‘tour d’horizon’ of some dynamical features in GR that point towards some sort of universal dissipation/smoothing mechanisms in the theory (Sect. 8.2.2), starting with the challenge of encoding prescribed initial data (Sect. 8.2.1), thus motivating an asymptotic reasoning approach. In Sect. 8.3, we will consider a specific astrophysical study case, namely binary black hole (BBH) mergers, where asymptotic methods can be applied in various degrees, unveiling a hierarchy of universal patterns. We conclude in Sect. 8.4, highlighting the fruitfulness of (various forms of) asymptotic reasoning for probing (various types of) universal patterns, possibly hinting at fundamental underlying features. An important lesson of this contribution concerns the epistemic and ontological relevance—in particular for time—of operational considerations about dynamics in the GR setting, where there is no canonical notion of temporal evolution. In the reminder of this section, we briefly set up the conceptual framework of the sort of asymptotic reasoning we shall be using, in particular the related notions of structural stability and universality (relying on Batterman, 2002).

8.1.1 The Conceptual Framework of Asymptotic Reasoning: Structural Stability and Universality

A basic tenet in our discussion is the focus on universal patterns—universal in the sense of being insensitive to fine-

tuned details (see below)—emerging in gravitational dynamics behaviours that play a key role in the predictive power of the theory as well as in the understanding of specific underlying phenomena. Such a view builds on the notion of 'asymptotic reasoning' introduced by Batterman (2002), namely an approach to a given theory that explicitly sacrifices precision and exactness by eliminating details, in order to make the underlying patterns apparent.

When such underlying patterns display an appropriate form of *structural stability*, they become universal patterns providing the substrate for a notion of *universality* in the theory. Asymptotic reasoning is therefore tightly connected to the notions of 'structural stability' and 'universality'. Structural stability has a precise technical meaning in dynamical system theory, but in a broader perspective it captures the idea that small perturbations (e.g., in the initial configurations or the dynamical equations) do not lead to qualitatively different behaviours of the system under consideration (where 'small' and 'qualitatively different' need to be characterised according to the context). Within the framework of asymptotic reasoning, structural stability may be an indication that physically distinct systems can display similar dynamical patterns, irrespective of their differences—that is: universal patterns. Batterman (2002, 13) considers that two general features are characteristic of universality:

1. "The details of the system (those details that would feature in a complete causal-mechanical explanation of the system's behaviour) are largely irrelevant for describing the behavior of interest.

2. Many different systems with completely different "micro" details will exhibit the identical behavior."

In a sense, structural stability (understood in the broad perspective mentioned above) guarantees that these "details" (seen as 'small' perturbations) do not affect the "behavior of interest": it thus provides asymptotic reasoning its epistemic power to identify universal patterns.[1] Asymptotic reasoning is usually realised in a regime of the theory aiming at focusing on a specific phenomenon and not in the full theory, by filtering away degrees of freedom of the theory not relevant for the phenomenon. From a technical perspective, this is typically characterised by certain limits of a small/large parameter, often entailing a gain in the mathematical tractability and simplicity in the appropriate range determined by the asymptotic parameter.

Physics and mathematics are rich in examples of asymptotic treatments, ranging from the semi-classical approach to quantum mechanics, the renormalization group, quasi-geostrophic equations in geophysical fluids for explaining turbulence and inverse cascades, to the Post-Newtonian series in general relativity and the theory of the rainbow, among many others. The key point is that such a focus on the structurally stable features of specific phenomena in 'filtered' (or 'coarse-grained') regimes do reveal universality aspects of the full theory. As Batterman puts it:

> "[...] the existence of [structurally stable] solutions– the principal laws or basic features that explain the universal phenomenon of interest–is a reflection of the fact that the process governing the behavior has stabilized in a very important way in the asymptotic regime where the details of the initial and boundary conditions play no significant role." (Batterman, 2002, 59)

In this contribution, we will argue that such stabilization (dissipation/smoothing) mechanisms play a crucial role in the operational articulation of GR as well as in the emergence of universal patterns, and that asymptotic reasoning provides a suitable framework to study these intriguing features of gravitational dynamics.

8.1.2 Spacetime Dynamics as an Asymptotic Limit: 'Operational' Gravity in Action

The physics of gravitation stands at a crossroads between 'theoretical' and 'operational' problems. On the one hand, as a cornerstone in our understanding of fundamental physics, gravity permeates the conceptual and structural developments of current theoretical physics. The quest of a theory of quantum gravity belongs to this category, but it is not the only setting in which gravity offers a structuring framework for fundamental physics (see Aldaya et al., 2011). On the other hand, with the advent of precision cosmology and gravitational wave physics, the physics of gravitation also plays a crucial 'operational' role both to guide the research and to interpret the observations.

However, there is an explicit tension between both perspectives. Whereas the operational side of gravitational physics explicitly relies on GR and hence seems to fundamentally require spacetime (e.g., curvature) notions, the very need for spacetime structures is under question on the theoretical side of (quantum) gravitational physics. But even if we stay at the (classical) spacetime level, there are tensions between what we may refer to as a fundamentally geometric perspective and the 'operational' demands of confronting astrophysical systems. The notion of mass in GR constitutes a paradigmatic example: whereas the geometric characterisation of mass involves the full spacetime under specific asymptotic conditions at infinity—and hence cannot be associated with a compact (in space)

region of spacetime[2]—the everyday practice in gravitational wave research assigns ‘effective masses’ (as well as linear and angular momenta) to individual compact objects such as black holes. This is not an academic point, but a very pragmatic issue that is the focus of intense research (e.g., the recent study by Rácz, 2024 to provide better local estimates for the angular momentum; see also Szabados, 2008; Jaramillo & Gourgoulhon, 2011 for reviews on the subject).

In this contribution we will focus on a spacetime description of gravitational dynamics, arguing for the fruitfulness of a form of asymptotic reasoning in understanding the emergence of universal patterns that are crucial to the operational side of gravitational physics. Indeed, asymptotic reasoning can be understood as a methodological strategy to address the concrete astrophysical (and cosmological) scenarios that are naturally identified as genuine temporal evolution problems. In this perspective, spacetime (Lorentzian) structures actually provide a sound framework to characterise such evolutions, in particular allowing non-essential degrees of freedom to be ‘filtered’ in a spirit akin to asymptotic reasoning. The remarkable fact we will discuss in the next section is that, although fundamental obstacles hinder the canonical (geometrical) construction of relevant astrophysical notions—such as (quasi-)local physical quantities mentioned above—a sort of structural stability smooths out the differences, giving rise to universal behaviours that guarantee the operational usefulness of different (and differing) effective estimates.

8.2 General Relativity: The Initial Value Problem and Universality

‘Dynamical change’ can be cast in different schemes in the theory of general relativity. A notion of ‘evolution’ as the continuous description of the physical system in ‘time’ is a natural approach when upgrading to the relativistic setting the picture provided by Newtonian celestial mechanics. Given that GR encodes (in geometrical terms) the dynamics of spacetime itself, without an explicit (and canonical) evolution formulation, it is a remarkable feature of the theory that it indeed admits a well-posed evolution description at all. In particular, GR admits a well-posed initial value formulation, that can adopt different realisations, as a hyperbolic (wave) partial differential equation (PDE) system subject to constraints or, alternatively, as in the 3+1 Hamiltonian version, each formulation highlighting a structural aspect of the theory. This initial value perspective will play an important role in the following, since it underlies aspects of the class of universal behaviours that we aim at discussing. However, it is important to emphasize that this is not the only option for exploring gravitational dynamics in general relativity. Schemes based on the notion of direct and inverse scattering, built on asymptotic data rather than detailed bulk quantities ‘evolving in time,’ have also played a role in the development of the theory, with a particular renewal in recent times in the setting of gravitational wave physics[3] (cf. e.g., point (vii) in Sect. 8.2.2 and points (iv) and (v) in Sect. 8.3.1). Both families of approaches—the ‘scattering approaches’ focusing on the initial-final states and the dynamical (‘on-the-fly’) evolution approaches—are intimately connected, but they stress different aspects of the theory. We will now focus on the continuous evolution schemes based on the initial value formulation of GR—although we will see that scattering approaches eventually enter the discussion to shed light on some universality aspects of the dynamics.

8.2.1 The Initial Value Problem in General Relativity

The initial value formulation is a very natural approach to study the dynamics of point particles in Newtonian dynamics, both from the perspective of the physical characterisation of the system and for the mathematical analysis of the dynamics. The latter involves both the aspect of describing the evolution of a given system, as well as the question of analysing its stability when varying initial conditions (namely when studying the impact of performing variations in the phase space). The same remains true when upgrading the theory from particles to fields, but still with a privileged global time providing a canonical notion of simultaneity and an instantaneous 'synchronisation' mechanism through some 'action at a distance.' In all these (non-relativistic) contexts, setting initial conditions at a given time is a natural and unproblematic procedure.

The situation is less clear in the context of a relativistic theory, in particular in theories of spacetime dynamics: whereas the mathematical question is well-posed and leads to important insights into the structure of the theory, from a physical perspective it is not given for granted that the initial value approach provides the natural scheme to study the dynamics of spacetime from a fundamental perspective (although it may prove very useful for a number of purposes).

The mathematical consistency of the initial value problem for 'well-posed' theories of spacetime dynamics is guaranteed in the following sense: prescribing appropriate initial data on an initial manifold Σ_0 fixes completely, up to the so-called *Cauchy horizon*, a spacetime (M, g_{ab}) where the manifold M is built up in terms of the slicing $\bigcup_t \{\Sigma_t\}$, where slices Σ_t are uniquely evolved from data at the initial slice Σ_0.[4] The well-posedness of such initial value problem for GR has been proven by the foundational results of

Choquet-Bruhat and Geroch (Fourès-Bruhat, 1952; Coquet-Bruhat & Geroch, 1969). In this context, one often refers to globally hyperbolic spacetimes, which are spacetimes (M, g_{ab}) admitting a so-called *Cauchy hypersurface* Σ_0 from which the spacetime can be reconstructed as a maximal future development of an initial data set.[5]

Data at the initial slice are provided by the pair (γ_{ij}, K_{ij}), where γ_{ij} is a Riemaniann metric in Σ_0 and K_{ij} is a symmetric tensor to be eventually interpreted as the extrinsic curvature of (Σ_0, γ_{ij}) into (M, g_{ab}). The pair (γ_{ij}, K_{ij}) must satisfy (in vacuum) the so-called constraint equations

$$R + K^2 - K_{ij}K^{ij} = 0 \tag{8.1}$$

$$\nabla_j(K^{ij} - \gamma^{ij}K) = 0\ , \tag{8.2}$$

where $K = \mathrm{Tr}_\gamma K_{ij}$, ∇ is the Levi-Civita covariant derivative associated with γ_{ij}, with Ricci scalar R. The first equation is referred to as the Hamiltonian constraint, whereas the second is the momentum constraint. Once these data are prescribed, different schemes exist in order to provide a detailed account of the evolution building a unique (maximal future development) spacetime, as well as a methodology to analyse the impact of changing the initial data (i.e., to probe the phase space of the system).[6] Therefore, in this mathematical sense, the situation is similar (and this is a remarkable non-trivial fact) to what we commented above for the initial data problem in Newtonian particle dynamics. A particular qualitative different feature in the general relativistic setting is that the 'time function' t slicing the spacetime, is constructed along the evolution process in connection with the Hamiltonian constraint (8.1), and is not unique.[7]

However, when considering the problem from a physical perspective, the situation departs from that of Newtonian physics in at least two important ways: first, in the very

sense given to the global time t, and second, in the ability to prescribe the physical content 'ab initio' at a given initial time.

Regarding the first aspect, in the spirit of the initial data strategy one should be able to set initial data in a Cauchy slice Σ_0 providing a particular notion of 'simultaneity' at an initial time $t = t_0$. However, this is operationally a challenge in a relativistic theory, and the very schemes one could devise to 'synchronise' data at a given (initial) time suggest that this is not the natural approach. There is simply no unique and natural notion of simultaneity, and there is no (operational) way of setting simultaneous data 'instantaneously' at spatially separated distances.[8] A smooth 'temporal function' t always exists, as shown in Bernal and Sánchez (2005), providing an ordering structure for data that are correlated (constrained) at a given constant time t, but it does not provide an actual physical magnitude that we could call 'time' in the sense that can be attributed to observers using clocks, although in a stably causal spacetime the function t is strictly increasing on any future-directed causal curve (Hawking, 1969).

Regarding the second aspect, a conceptual and technical challenge actually arises when prescribing the pair (γ_{ij}, K_{ij}) , aiming at encoding the physical content of the system one wants to study. The difficulties of this prescription step are ultimately rooted in a form of 'non-locality'[9] and the non-linearity of the PDE systems obtained from the geometric constraints (see the detailed discussion in Jaramillo & Lam, 2021). In particular, prescribing individual masses, angular momenta and other physical quantities of various compact objects interacting dynamically is a question not admitting a single canonical answer (Szabados, 2008; Jaramillo & Gourgoulhon, 2011). Different prescriptions do exist (except for global quantities, typically at infinity for

isolated compact systems) and all of them provide a consistent picture, as long as we keep track of their correspondences when comparing initial data. In addition, because of the nature of the problem to be solved for constructing consistent initial data, it is very difficult not to include 'junk' initial data that is unrelated to the system under consideration.[10]

But the theory displays a remarkable feature: the resulting dynamical system is quite insensitive to the variations among different prescriptions for encoding some physical content (e.g., corresponding to some observational data) and, more importantly, it efficiently eliminates the features associated with the junk part of the initial data (cf. point (viii) in Sect. 8.2.2). A dynamical spacetime 'dissipation/smoothing mechanism' occurs, coarse-graining the dynamics and dissipating the tiny differences between alternative effective descriptions of the same physical system. Such a 'smoothing out' phenomenon can thus be understood as an instance of structural stability, and, given the difficulty to encode sharp prescribed features in the initial data, it turns out to be crucial for the predictive power of the theory (Jaramillo & Lam, 2021).

In summary, we retain two important points from the discussion above about the initial data ('evolution') perspective on spacetime dynamics:

(i) Spacetime dynamics incorporates a dissipation/smoothing mechanism that efficiently coarse-grains over (a sector of) the dynamical degrees of freedom, smoothing out their evolution.

(ii) A smooth, non-unique time function t can be constructed permitting an initial value formulation of spacetime dynamics in which a spacetime dissipation/smoothing mechanism guarantees the structurally stability of the initial value scheme by

smoothing out non-essential ‘sharp’ features (e.g., the junk part of the initial data). The function t does not correspond to any actual observable to be measured by physical clocks,[11] but rather constitutes a tool to parametrise ‘continuous change’ (evolution): for instance, it enables the encoding of correlations between different events in spacetime in terms of the system of constraint equations.

8.2.2 Spacetime Dissipation/Smoothing Mechanism

Although the effectiveness of the underlying dissipation/smoothing mechanism is crucial in the common application of general relativity to relativistic astrophysics, its precise nature is difficult to pinpoint and remains rather elusive. It is at this point that asymptotic reasoning proves fruitful, by considering different asymptotic regimes and limits of the theory and unveiling the underlying patterns.

We list below some instances illustrating (various forms of) this dissipation/smoothing mechanism, providing simpler dynamics than the one expected a priori (and focusing on compact objects in general relativity)[12]:

(i). *Post-Newtonian ‘unreasonable effectiveness’*: in the description of the two-body problem in general relativity, the Post-Newtonian formulation provides an asymptotic expansion describing the regime in which both velocities and self-gravity of compact objects are commensurate and small, namely aiming at modelling the ‘inspiral phase’ in compact binaries. The surprising and unexpected fact is that the scheme continues to provide an excellent approximation beyond its limits of validity (Will, 2011), well into the fully non-linear ‘merger phase’. The binary dynamics is simply smoothed-out at the merger phase.

(ii). *Effacement property or 'Effacing Principle'* (Damour, 1987): still in the context of the inspiral phase of binaries, a remarkable simplification occurs in the dynamics, namely the fact that the internal structure of the compact object starts to be accessible only at at high order in the expansion and it is effectively hidden till late stages when tidal effects start playing a role. This translates in the fact that compact objects can be just parametrised by mass M and angular momentum J. This effacement of the internal structure is associated with the (strong) equivalence principle in general relativity (Will, 2014).

(iii). *Effective one body approach to binaries*: in addition to these features, further elements in the Post-Newtonian description converge to build a remarkable simple picture in which the full relativistic dynamics is captured in terms of the dynamics of a single body under an effective potential, namely passing from an in principle complicated non-linear PDE to a much simpler ordinary differential equation picture (Damour & Nagar, 2016).

(iv). *Close-limit approximation and effective linearity*: the so-called 'close-limit' approximation of BBH mergers introduced by Pullin and Price (1994) shows that the merger phase is very well captured by the perturbation of a single black hole. On the other hand, perturbation theory provides an appropriate mathematical setting to describe the late time ringdown phase. In this sense, the 'close-limit' approximations supports the validity of the ringdown perturbative treatment beyond its a priori

range of validity. This is complementary, from the post-merger side, to the 'unreasonable effectiveness' of the Post-Newtonian scheme in the inspiral pre-merger phase. In sum, the whole binary dynamics including the 'non-linear' merger phase is 'smoothed-out' giving rise to a much simpler picture than expected and implementing an 'effective linearity' (see discussion in Jaramillo, 2022).

(v). *Gravitational collapse, cosmic censorship and 'no hair' theorem*: the picture of gravitational collapse in classical general relativity is constituted by a series of theorems and conjectures providing an interesting example of asymptotic reasoning. Starting from Penrose singularity theorem (Penrose, 1965) guaranteeing the formation of a singularity under very generic conditions (and crucially oblivious to details), the cosmic censorship conjecture (Penrose, 1979, 1982) then proposes the formation of a horizon causally disconnecting the singularity from a distant observer. Under a second (stability) conjecture, general relativity dynamics drives this isolated system to stationarity.[13] Finally, under the assumptions of stationary, vacuum and a (Killing) horizon, the black hole uniqueness theorem (see e.g., Chruściel et al., 2012) establishes that the resulting compact object is a (sub-extremal) member of the Kerr spacetime family. This implements a black hole 'no hair' result where the final state only depends on two parameters, namely the mass M and angular momentum J of the black hole. This resonates with Damour's effacing property, in a picture where dissipation (at infinity and at the horizon) as well as the formation of a positive mass (stable) self-gravitating compact object play key roles. The whole framework

instantiates a dramatic example of asymptotic reasoning where a universal pattern emerges (in the late time asymptotic limit), independently of any fine details of the initial data or the dynamics.[14]

(vi). *Redshift effect and stability*: a crucial element in the dynamical picture above is the 'red-shift effect' (Dafermos & Rodniaski, 2009, 2013) that provides a stabilizing mechanism for the wave dynamics in black hole backgrounds. In particular, the role of the black hole horizon as a dissipation element becomes explicit through the 'local redshift effect.' The latter provides the key geometric property that controls the strongly dissipative character of black hole quasi-normal modes, by structuring the quasi-normal frequencies in the complex plane into bands of width given by the surface gravity κ (Warnick, 2015). This guarantees the fast decaying time-scales of black hole overtones, as well as controlling the universal Weyl asympotics of black hole quasi-normal modes (Jaramillo et al., 2022).

(vii). *Spacetime dynamics 'cross-correlation' effectiveness*: the approach to spacetime dynamics (Jaramillo, 2011; Jaramillo et al. 2011, 2012a, b) based on the correlations between geometric quantities at an outer (e.g., null infinity) and inner (e.g., black hole horizon) 'screens' strongly relies also on an effective (approximate) linear dynamics of the propagating gravitational fields in the spacetime bulk. Why such a linearity assumption holds in the appropriate regime is not clarified in this scheme.

(viii). *Numerical relativity initial data and 'junk radiation'*: constructing initial data numerically in a initial value problem for spacetime dynamics involves the

numerical resolution of the constraint equations. Loss of accuracy in such a resolution 'contaminates' initial data generating what is known as 'junk radiation.' However, such junk radiation is typically very effectively radiated away leaving a consistent filtered constraint-preserving system.[15] This represents a practical instance of asymptotic reasoning in action, relying on a dynamical mechanism that again exploits an effective linearity.[16]

(ix). *Geometric optics approximation and high-frequency limit in general relativity*[17]: general relativity is indeed a very special theory, not just any spacetime dynamics theory. Although it is a non-linear PDE theory, non-linear terms enter the picture in a very special way, which, in combination with the (diffeomorphism) gauge invariance, makes the theory effectively more linear than a priori expected. This linearisation phenomenon is explicitly seen in the geometric optics approximation of general relativity, namely the asymptotic limit of high-frequencies. Specifically, wave dynamics in the geometric optics scheme gives rise to a set of transport equations for wave profiles that, in the weakly non-linear regime, generically include non-linear terms. However, in the case of Einstein equations, a special phenomenon[18] occurs and non-linear terms finely compensate, rendering the transport equation linear even in the (weakly) non-linear regime. This remarkable feature of general relativity (in particular preventing waves in spacetime dynamics from breaking as sea waves do on the coast) was identified by Choquet-Bruhat (1969, 2008), and recently further systematized and extended in

Touati (2023a, b).[19] In the very spirit of the asymptotic reasoning pursued here, this suggests the possibility that this effective linear universal pattern identified in a (geometric optics) asymptotic limit actually corresponds to a structural underlying feature of the full theory. This would provide a non-trivial insight into the various instances of 'effective linearity' presented above.

(x). *Universality and simplicity of the BBH merger waveform*: in Jaramillo and Krishnan (2022a, 2022b) and Jaramillo et al. (2023) a scheme based on asymptotic reasoning is proposed to address the unexpectedly simple (and universal) structure of binary merger gravitational waveforms. This will be sketched in Sect. 8.3.

The previous points suggest the presence of an underlying mechanism in GR that smooths out small details, providing a substrate for structural stability and hence for universal behaviours in a GR regime relevant to astrophysics (and cosmology). This mechanism seems to provide a commensurate balance between efficient dissipation[20] and the realisation of the (strong) equivalence principle in self-gravitating objects, mediated by an enhanced effective linearity in wave propagation.

8.3 Binary Black Hole Mergers and Universality: A Study Case

In the previous section we have presented general relativity as an initial value problem and have argued that a type of structural stability, involving a dissipation/smoothing mechanism, plays a crucial role in building the concrete predictive power of the theory. Such a perspective relies on

an asymptotic reasoning scheme, focusing on specific phenomena through the elimination of non-essential details. In this sense, and in order to complement the instances of the dissipation/smoothing mechanism presented in Sect. 8.2.2, we focus in this section on a specific phenomenon, namely the point (x) above concerning the binary merger waveform.

The dynamics of compact object binaries and its associated gravitational wave emission, displays three phases: (1) an inspiral phase where the two compact objects orbit each other and slowly approach as they lose energy by gravitational wave emission, (2) a merger phase, dominated by the non-linear aspects of the theory, where the two compact objects coalesce into a single compact object and, (3) a relaxation phase where the remnant rings down and approaches stationarity in a regime controlled by linear perturbation theory. Early educated expectations (cf. e.g., Kip Thorne's cartoon in Fig. 1 of Schutz, 2005) suggested an initial sinusoidal signal with slow amplitude increase in the inspiral phase, followed by a violent merger phase (a 'spacetime storm') and ending at the late ringdown in an exponentially damped sinusoidal signal determined by its characteristic (complex) quasi-normal frequencies. Although the inspiral and the ringdown were finely captured by the Post-Newtonian and perturbative treatments, respectively, the merger phase expectations completely failed to match the obtained numerical results first (Pretorius, 2005), and the gravitational wave observations later (Abbott et al., 2016): in deep contrast with the expected violent features, the merger waveform presents a smooth and simple interpolation between the inspiral and the ringdown. And the signal is not only simple, but also universal in the sense of being largely independent of initial data (see the discussion on universality in Sect. 8.1). At the sight of this unexpectedly simple behaviour, a natural question raises:

Is the binary black hole merger waveform boring or elegant?
Attitudes towards this question typically differ according to the background and interests of the researcher. Whereas astrophysicists tend to see such a signal as 'boring,' especially when comparing it with the 'rich' signal associated with neutron star mergers (much more complex due to the matter equation of state), theoretical physicists tend to see it as 'elegant,' its simplicity suggesting an underlying mechanism in the theory that prevents complex behaviour from occurring. Here we adopt this latter approach, namely, the view that the simplicity and universality of the signal provide a probe into the fundamental structures of (a sector of) the theory.

8.3.1 Asymptotic Reasoning in the BBH Merger Waveform: From Universality to Integrability

In Jaramillo and Krishnan 2022a, b and Jaramillo et al. (2023) a bottom-up approach is proposed to explore the simplicity and universality of the signal in an attempt to identify the underlying structurally stable mechanisms. This effort is structured in a hierarchy of layers, each one providing a (horizontal) instance of asymptotic reasoning and the full hierarchy providing itself a (vertical) instance of asymptotic reasoning, since each new layer adds some details filtered in the previous one. The asymptotic reasoning hierarchy is displayed in Table 8.1, where for each model the mathematical/physical framework is presented, as well as the underlying key structures and mechanisms.

Table 8.1 Asymptotic reasoning hierarchy of models for the BBH merger dynamics (from Jaramillo et al., 2023)

Asymptotic BBH Model	Mathematical/Physical Framework	Key Structures/Mechanisms
Fold-caustic model	Geometric Optics	Arnol'd-Thom's Theorem
	Catastrophe (Singularity) Theory	Classification of Stable Caustics
Airy function model	Fresnel's Diffraction, Semiclassical Theory	Universal Diffraction Patterns in Caustics
Painlevé-II model	Painlevé Transcendents and Integrability	Painlevé property
	Self-force calculations and EMRIs	Non-linear Turning Points
KdV-like model	Inverse Scattering Transform and Integrability	Painlevé test, Lax pairs
	Dispersive Non-linear PDEs	Darboux transformations, Soliton Scattering
	Critical Phenomena in Dispersive PDEs	Universal Wave Patterns, Dubrovin's Conjecture
Propagation models on (anti)-Self-Dual backgrounds	Ward's Conjecture and Integrability	(anti-)Self-Dual degrees of freedom, Instantons, Tunneling
	Twistorial techniques	Penrose Transform, 'Twistor' BBH data

In the following we sketch the ideas underlying this hierarchy and refer to Jaramillo and Krishnan 2022a, b and Jaramillo et al. (2023) for more details:

(i) *Gravitational wave detection as a caustic in time*: this presents a minimalistic model for detection, by considering a geometric optics scheme in which gravitational radiation is modelled by rays. In such a framework, caustics, that can be understood as the enveloping of rays, are generic and, once formed, structurally stable (Arnol'd-Thom theorem). The important point is that 'in the interior of the caustic' the intensity of the signal increases as the caustic is approached, diverging at the caustic and then dramatically falling down (or disappearing in the case

of the so-called fold caustic). This minimalistic model proposes that a caustic phenomenon, occurring in time and controlled by the 'fold caustic' generating (phase) function $\phi_{\rm fold}(t,s) = ts + \frac{1}{3}s^3$, with s a 'state variable' characterising the system, underlies the signal detected by the interferometer located at some 'fixed space point': the detector receives a signal whose intensity growth with a Post-Newtonian pattern correctly captured by the fold caustic $\phi_{\rm fold}(t,s)$ properties, diverging at the coalescence and fully vanishing afterwards, so that there is no hope to include the ringdown. The merits of this basic model are to clean up almost all details except a minimalistic description in terms of rays, to be universal thanks to caustic structural stability and to provide the skeleton for the next diffraction model.

(ii) *Caustic diffraction patterns and Airy function BBH waveform model*: the second layer adds a new physical ingredient beyond rays, namely it incorporates the wave-nature of the radiation but without resorting to the full theory, but only keeping the minimal wave elements in the form of diffraction theory. In a slogan form: if caustics provide the skeleton, diffraction provides the flesh (Berry, 1976). On the one hand, this regularises the caustic divergence and, on the other hand, it provides a universal and structurally stable (caustic) diffraction pattern. The fold caustic leads to the so-called Airy function

$$h(t) \sim {\rm Ai}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds^{i\phi_{\rm fold}(t,s)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds e^{i(ts+s^3/3))}. \quad (8.3)$$

This extremely simple Airy model (Jaramillo & Krishnan, 2022a) for the gravitational signal $h(t)$ detected at the gravitational wave antenna captures

remarkably well the binary-merger phases. From the asymptotic reasoning perspective, the universal pattern that is unveiled in this high-frequency asymptotic limit is the Airy function. More specifically, the underlying Airy equation, namely $\ddot{h} - th = 0$, is the actual universal pattern that is identified, with its structural role as a 'turning point' equation interpolating between an oscillatory (inspiral) regime and an (over-)exponentially damped one after the merger, together with its built-in intrinsic dispersive character.

(iii) *Painlevé-II turning point model and EMRIs*: the Airy function does not allow to account for the transition to the ringdown phase, a non-linear term being needed to account for the behaviour of the frequency at the coalescence time. Pushing along the turning-point aspect, the archetype of non-linear turning point generalisation of the Airy equation is the so-called Painlevé-II equation (Ablowitz & Segur, 1977), namely $\ddot{u} - tu - 2u^3 - \alpha = 0$. A remarkable feature is that such an equation plays indeed an explicit role in the exact solution of the self-force problem of a charged particle in a Coulomb field (Rajeev, 2008), suggesting its relevance for the BBH problem in the asymptotic limit provided by extreme mass ratio inspirals (EMRI), as suggested in Jaramillo and Krishnan (2022b). The universal pattern extracted from this asymptotic limit would then be the Painlevé-II equation. But a crucial remark needs to be made here: Painlevé-II and more generally the six Painlevé transcendents (Clarkson, 2003; Conte & Musette, 2008) are a smoking gun of an *integrable* structure. The universal pattern identified at this level is the Painlevé-II structure and its associated integrability, which threads the BBH dynamics from the inspiral to

the ringdown, including the intermediate transient merger, and which grounds its universality and simplicity.

(iv) *Non-linear dispersive PDE integrability, universal wave forms and 'soliton resolution'*: in the next asymptotic level, new elements are incorporated in the model to upgrade Airy/Painlevé-II ordinary differential dynamics to a PDE description. A crucial methodological step is enforced at this stage, aiming at reconciling the success of the effective linear dynamics referred to above (see the points (iv), (vii), (viii) and especially (ix) in Sect. 8.2.2) with the non-linear character inherent to Painlevé-II integrable systems and the associated universal wave forms appearing in such non-linear dispersive settings (Miller, 2016). We adopt a 'wave-mean flow' perspective on dynamics by separating fast and slow degrees of freedom:

(a) Fast degrees of freedom with effective linear dynamics, propagating on a slowly dynamical (non-linear) background and reaching infinity where the BBH waveform is observed.

(b) Slow degrees of freedom subject to a non-linear (quasi-)integrable dynamics instantiating the Painlevé-II structure and providing the background for the fast degrees of freedom.

Integrability is therefore not upgraded to the full general relativity theory, but only to a (slow) sector of it. The claim that integrability only affects a sector of the theory is crucial and has important consequences. On the one hand, it is not clear that in this 'wave-mean flow' scheme the dynamics should be fully treated in the 'initial value' formulation detailed in

Sect. 8.2; rather, an appropriate combination of initial value and 'direct/inverse scattering' approaches may be more appropriate (see e.g., Ashtekar et al. 2022a, b; Strominger, 2018). Scattering approaches often take the best out of integrability, in particular by identifying hidden symmetries in the system, such as the Darboux symmetries in Lenzi and Sopuerta (2021a, b, 2023a, b, 2024). On the other hand, the '(linear) fast/(integrable) slow' split offers an asymptotic methodological avenue to probe a dynamical feature that probably underlies the full theory and is known as the 'soliton resolution' conjecture (Bizón et al., 2022): generic global-in-time non-linear wave dynamics tends to decouple universally at late times into 'solitons' plus 'radiation.' This behaviour constitutes an archetype of asymptotic reasoning, since the initial details are completely erased in the process, only remaining the final radiation scattering over 'solitons', namely special attractor solutions in the phase space of the theory. In this sense, the gravitational collapse picture in point (v) of Sect. 8.2 provides an archetype of soliton resolution in the general relativity gravitational context.

(v) *Scattering in non-linear (integrable) self-dual backgrounds*: the previous layers have built a bottom-up asymptotic reasoning hierarchy for BBH dynamics by progressively adding new details, leading to a picture of BBH dynamics in which fast degrees of freedom propagate according to linear wave dynamics over a non-linear (dispersive) integrable PDE system. At this point, it is natural to wonder if this fast/slow '(linear) wave-(integrable) mean flow' picture can be reached from the full Einstein equations via a top-down approach in which details

are actually filtered out, rather than added. It is in this setting that the so-called Ward conjecture (Ward, 1985; Ablowitz et al., 2003) enters the picture, by providing a link between integrable systems and self-dual solutions in Yang-Mills and GR theories. More specifically, in the spirit of asymptotic reasoning, one would aim at starting from Einstein equations in the BBH setting and filter non-essential elements in order to cast the dynamics in terms of the scattering of fast degrees of freedom over 'bound-states' of the self-dual sector of the theory. Two remarks are relevant here: (a) self-dual integrable solutions in GR are either complex or are real solutions in the Euclidean counterpart of (Lorentzian) GR, that is, we are naturally led to address BBHs as a scattering problem over gravitational instantons; and (b) twistor theory is tailored to deal with self-dual structures in GR, making twistors the natural concept to characterise the phase space of integrable slow degrees of freedom in the BBH problem.

8.3.2 Fundamental Gravitational Structures from Asymptotic Reasoning

In the previous subsection we have presented a 'study case' to illustrate how, starting from a particular phenomenon (in our case the simplicity of the BBH merger waveform), one can use an asymptotic reasoning to probe the fundamental structures of the underlying theory, by unveiling universal patterns in an asymptotic limit. The claim is that such universal patterns reveal, beyond that particular asymptotic limit, a structure that may actually underlie the full theory well beyond the asymptotic limit where it becomes apparent.

Specifically, the previous BBH case suggests the following structural elements underlying the full theory:

(i) *Integrability of a sector of the theory: Painlevé-II structure*. The universality of the BBH merger waveform would be a manifestation of an underlying integrable sector of GR. Integrability is not claimed for the whole GR, which is known not to be integrable, but for the sector of the GR phase space associated with (solitonic) 'background' degrees of freedom. From a technical point of view, Painlevé-II transcendents would play a key role in this kind of dynamical transients.

(ii) *Hidden symmetries in the bulk spacetime*. Intimately associated with this integrability controlling soliton backgrounds, a set of hidden symmetries can underlie the dominating structural features of the dynamics in the bulk (in the spirit of Darboux' transformations identified by Lenzi and Sopuerta (2021a, b, 2023a, b, 2024) and intertwined with infinite-dimensional Lie algebras in the Korteweg-De Vries hierarchy). How these bulk symmetries interplay with asymptotic symmetries is a topic of current research pioneered by Lenzi and Sopuerta.

(iii) *Twistor and self-dual structures*. A most surprising outcome of the asymptotic reasoning discussion of the BBH merger waveform universality is the proposal that "twistor data" could parametrize the self-dual sector of the theory controlling the non-radiative degrees of freedom of the theory, with a version of the Penrose transform potentially providing an integral representation of such slow background solutions. This is complementary to recent approaches to study perturbations on self-dual backgrounds (Adamo et al. 2021, 2022a, b), although our scheme is a more humble one not attempting to provide a comprehensive formalism, but just aiming

at identifying the fundamental nature of the relevant degrees of freedom in the problem.

(iv) *Time and times*. As we have argued at the beginning of Sect. 8.1, the notion of time critically depends on the physical theory that is considered (as well as the way in which time is encoded in this theory). In our discussion, time has entered the initial value problem of GR as a 'temporal function' that permits to label spatial slices, providing a methodological tool to parametrise the transition between 'initial' and 'final' states of the system. It is unclear that such a function t represents a quantity that can be associated with what the 'clock' of an observer would measure, and it rather identifies sets of points in spacetime that are subject to (Einstein) constraints. But such a parametrised description has been key to unveil underlying structures of the theory, in particular through a methodological separation into fast/slow degrees of freedom, suggesting that the notions of time/clock are not necessarily unique, even in the same dynamical framework. Furthermore, the slow degrees of freedom might be better described by a scattering picture, where the continuous time description ('on-the-fly') is not even needed. The bottom line is that, in our initial value discussion, time appears as an epistemic rather than an ontological notion.

8.4 Beyond Spacetime Dynamics: Insights from Universal Patterns

The various examples discussed in Sect. 8.2.2 as well as the BBH merger study case in Sect. 8.3 show the fruitfulness of (the various forms of) asymptotic reasoning in the general

relativistic context, in particular when it comes to unveiling certain universal patterns. Of course, each example of Sect. 8.2.2 would require a detailed treatment—in this sense, our work here is very much programmatic. Identifying and understanding these universal patterns as well as the underlying (dissipation/smoothing) mechanisms connect to many important foundational topics in the theory of general relativity (as the examples of Sect. 8.2.2 testify)—and possibly beyond. Indeed, to the extent that these universal patterns are 'insensitive' to the underlying 'details' of a possible more fundamental (e.g., quantum gravity) theory, they may provide fundamental insights into dynamics and time in the gravitational context—whether or not (space)time turns out to be fundamental or emergent in some functional sense, according to a more fundamental description (Lam & Wüthrich, 2018, 2021). The dissipation/smoothing mechanisms that the different GR features discussed in this contribution seem to point at constitute a striking example.

As we mentioned in Sect. 8.1.1, the conceptual framework of asymptotic reasoning, whose fruitfulness in GR we are arguing for, is directly inspired by the work of Batterman (2002). However, we are not committed to his argument that universal features constitute evidence for a form of emergence and against (e.g., Nagelian) reduction. In fact, following Butterfield (2011a, 2011b), universality and emergence need not stand in opposition to (Nagelian) reduction: the idea is that emergence can be understood in terms of novel and robust behaviour (relative to some comparison class), which need not be incompatible with reduction. Similarly, we consider neither the (forms of) asymptotic reasoning nor the universal patterns discussed here in the GR context to stand in tension with reductionism. On the contrary, our main motivation is to probe fundamental structures in the theory by adopting an asymptotic reasoning methodology (as explicitly illustrated

in the integrability proposal from the observed universality in BBH waveform mergers in Jaramillo and Krishnan (2022a, 2022b); Jaramillo et al., 2023). This can be realised in complementary bottom-up (first sequence) and/or top-down (second sequence) approaches:

Asymptotic reasoning $\rightarrow$ Universal patterns $\rightarrow$ Fundamental structures,

Universal patterns $\leftarrow$ Asymptotic reasoning $\leftarrow$ Fundamental structures.

But much more work remains to be done. Indeed, it is important to stress that the various examples discussed in Sects. 8.2.2 and 8.3 refer to various forms of asymptotic reasoning, as well as to various types of universal patterns, possibly underlain by a variety of what we have called 'dissipation/smoothing mechanisms.' For instance, compare the forms of asymptotic reasoning at work in gravitational collapse (point (v) in Sect. 8.2.2) with the geometric optics approximation (point (ix) in Sect. 8.2.2), or compare the bottom-up with the top-down approach in the asymptotic reasoning hierarchy for the BBH merger waveform (Sect. 8.3.1). Both the variety and the common features call for further investigations.

We would like to end with two interrelated comments. First, part of the foundational considerations in this contribution arise from the operational side of GR, as the BBH merger study case testifies: casting the theory in terms of an initial value problem, running numerical relativity experiments and making sense of (the simplicity and universality of) the observed BBH merger waveform highlight the relevance of (various forms of) asymptotic reasoning and of underlying dissipation/smoothing mechanisms, which, as we have argued, deeply connects to foundational features of GR and beyond. More broadly, this highlights the relevance of epistemic considerations—sometimes of quite a pragmatic nature—at the non-fundamental level of GR for foundational (and ontological)

issues about (space)time. One important lesson here is that there is still much to be learned about spacetime at the level of classical GR and its operational articulation—when it comes to understanding the nature of spacetime, it is not all about quantum gravity.

This leads us to our second point: asymptotic reasoning in the GR context does not merely amount to a set of nice mathematical methods—e.g., allowing to enhance mathematical tractability and physical operationalisation (very important tasks in themselves!)—but also fundamentally involves what has been called an "exploratory role" in the literature (see section 4 in Shech, 2018 as well as references therein). As Shech puts it (writing about "infinite idealizations," to which asymptotic reasoning belongs): "it is suggested that infinite idealizations may play an indispensible role in the exploration of scientific theories, intertheoretic relations, and phenomena, and that possibly such exploration is needed for deep sense of scientific understanding" and that, in addition, "infinite idealizations do seem to play an indispensible role in exploring the possible representational structure and foundations of scientific theory" (Shech, 2018, 11)—here the theory of general relativity.

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Footnotes

1 On the epistemological relevance of structural stability, see the discussion in Fletcher (2020).

2 For a philosophy of science discussion, see Lam (2011).

3 See e.g the cross-correlation approach to strong-field spacetime dynamics proposed in Jaramillo (2011), Jaramillo et al. (2011, 2012a, 2012b) and further

developed in works such as Gupta et al. (2018), Prasad et al. (2020) and Iozzo et al. (2021), the 'gravitational wave tomography' program in Ashtekar et al. (2022a, 2022b), or some of the elements in the celestial holography framework (Strominger, 2018).

4 Here and in what follows, 'completely' and 'unique(ly)' should be read 'up to diffeomorphisms'—which does not affect our argument.

5 Note that not all spacetimes admit such a Cauchy hypersurface, and therefore not all spacetimes are a priori accessible as evolution of appropriate initial data. In this setting, the notion of strong cosmic censorship (e.g., Andersson, 2004) aims at ensuring that spacetimes cannot be extended beyond their *Cauchy horizon*, namely the boundary of the maximal future development. Globally hyperbolic spacetimes are thus the only ones relevant from an evolution point of view.

6 A particularly enlightening description of this aspect is the account in Fischer and Mardsen (1972), where the ADM version of Einstein equations is set as a dynamical system in a phase space, and the dynamics is explicit written in terms of the coadjoint of the differential of the constraint mapping. This underlines the crucial role of Eqs. (8.1)–(8.2), not only to constraint initial data, but also to determine the dynamics.

7 It is also a highly non-trivial feature of globally hyperbolic spacetimes that they do admit a 'temporal function' t that is not only continuous but differentiable (Bernal & Sánchez, 2005). If such a function does not provide a unique, global time, it does however play a key structuring role in the theory.

8 More precisely, whereas such an operational procedure to set a priori 'simultaneous' data is lacking for the Cauchy slices here considered, it can indeed (and remarkably) be devised for the so-called "hyperboloidal slices" (cf. (Zenginoglu 2024) for a detailed discussion). This hints to the fundamental structural role of hyperboloidal foliations in GR spacetime dynamics, as advocated by Zenginoglu (2024).

9 In the standard approaches to solve the constraints, this 'non-locality' stems from the elliptic nature of the problem on Σ_0; other (e.g., parabolic) approaches

can however be envisaged, see Jaramillo and Lam (2021) for a discussion.

10 Following Jaramillo and Lam (2021), we stress that these difficulties in constructing prescribed initial data have an epistemic rather than ontological nature: along the evolution of any system building up a dynamical spacetime, the constraint Eqs. (8.1)–(8.2) are identically satisfied at each (arbitrary) spatial slice $\sum$, since geometrically they correspond to the Gauss-Codazzi equations guaranteeing that slices build up a spacetime. The difficulties mentioned here are related to the epistemic need to construct initial data according to a given prescription at a given (finite) initial time. See however point (viii) in Sect. 8.2.2.

11 It is though a growing function along the trajectory of any future observer (Hawking, 1969).

12 In a cosmological setting, one could wonder if the homogeneity at large scale could respond also to a dissipation/simplification mechanism. This will be addressed elsewhere.

13 The positive mass theorem suggests that, under 'physically reasonable' dynamics, an isolated system, in which no energy is injected and dissipation occurs at infinity and at the horizon, should reach a stationary regime. But what is this specific physically reasonable dynamics is not clarified at this stage, constituting one of the main open issues in mathematical relativity, namely the complete proof of black hole stability.

14 The role of 'universality' and 'robustness' when semi-classical effects are incorporated in this black hole setting, giving rise to the so-called Hawking radiation (Hawking, 1975), has been systematically addressed by Gryb et al. (2021). The authors conclude that the involved 'trans-Planckian' problem, that threatens the inner consistency of the semiclassical treatment in this black hole setting, is not appropriately addressed by any of the proposed universality arguments, in stark contrast with the successful Wilsonian arguments in the context of condensed matter. This criticism does not affect our classical discussion, that can be seen as an instance of 'soliton resolution' (Bizón et al., 2022) in the gravitational setting (see point (iv) in Sect. 8.3.1).

15 For a review of the issues at the interface between numerical and mathematical general relativity, see e.g., Pfeiffer (2005) and Jaramillo et al. (2008).

16 We note that, as pointed out by their promoters, the practical application of the 'gravitational wave tomography' program (Ashtekar et al. 2022a, b) to numerically constructed spacetimes relies, at least to a certain extent, on such efficiency to eliminate junk radiation.

17 We are fully indebted to Arthur Touati for his patient and detailed explanations on this point. We owe our insights here to him, although misinterpretations of the mathematical results are our sole responsibility.

18 Specifically, denoting formally by g the spacetime metric, semi-linear terms $\sim (\partial g)^2$ can be absorbed into quasi-linear terms $\sim g\partial^2 g$, that can then be eliminated by an appropriate choice of (generalised harmonic) gauge. This is a very special feature of general relativity. Interestingly, this resonates with the discussion in Gasperin and Jaramillo (2022) about an effective virialization phenomenon in the appropriate regime of general relativity.

19 The phenomenon here discussed is known as *transparency* in the geometric optics setting. Touati (2023a) provides a sounder basis to the geometric optics treatment of general relativity in Choquet-Bruhat (1969, 2008), in particular applying it to prove a version of Burnett's conjecture (Burnett, 1989) in the spacetime high-frequency limit.

20 In particular, dissipation occurs at the spacetime 'boundaries', including at horizons when black holes are present. This means that the dissipation/smoothing mechanism that we are discussing is in principle restricted to the causally accessible outer domain, therefore not including black hole interiors. Non smooth GR phenomena such as spikes in the setting of the BKL conjecture are therefore not affected by this mechanism. Cosmic censorship seems to play a crucial role for this particular discussion of asymptotic reasoning in the gravitational context.

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9. On the Problem of Time in Asymptotically Safe Quantum Gravity

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Abstract
Quantum gravity has often been approached by scholars, as well as presented to the general public, as a very special theoretical subject, which needs a peculiar treatment, starting with the resolution of its foundational issues. Among the latter, the problem of time, also called the problem of reparametrization invariance, has often been particularly highlighted. Other notable and related challenges are the question of emergence or fundamentality of spacetime, and the construction of a space of states, of a probability interpretation, of a complete cosmological evolution. Nonetheless, a particular line of research called asymptotically safe quantum gravity still tries to make sense of the topic by quantizing general relativity on the same footing as any other field theory. Despite apparently ignoring or avoiding a direct

confrontation with the above-mentioned foundational problems, this formulation of quantum gravity leads to a definite view on them. This contribution provides a brief introduction to the asymptotic safety scenario for quantum gravity, especially focusing on the role that the problem of time and its related foundational questions have within this formulation. We review the recent developments and proposals towards some of the ultimate quantum-gravity research goals: background invariance, unitarity, cosmology and the origin of scales.

9.1 Introduction: What Is Asymptotic Safety?

Asymptotic safety is definitely a newcomer in the kindergarten of quantum gravities. Despite the first explicit statement of the idea dates back to an early exploratory work by Stephen Weinberg (Weinberg, 1978; 1980), the actual birth of a scholarly community around this idea started much later, following two important articles. The first rather technical work was by Martin Reuter (Reuter, 1998). No explicit mention of the asymptotic safety program by Weinberg was in it. The second article was by the PhD student Wataru Souma, had a clear reference to asymptotic safety already in the title and abstract, and was based on the application of the techniques and considerations developed in Reuter's article (Souma, 1999). After these articles, in the first decade of its history, the asymptotic safety community had been rather sparse but nevertheless managed to collect a wealth of results drawing a first portrait of the proposed scenario. The following decades saw a rapid growth of the community. For a detailed history of these developments we refer the interested reader to more in-depth reviews (Percacci, 2017; Reuter & Saueressig 2019; Bonanno et al. 2020). But what

precisely is this proposal? In first approximation I would convey the asymptotically safe quantum gravity (ASQG) proposal to a non-specialists audience as follows: it is possible to quantize general relativity as any other field theory, by means of a variety of methods (canonical quantization, covariant quantization, perturbation theory, discretized approaches and more) and still get a single consistent description, that is, the same set of predictions for physical experiments; all this thanks to the phenomenon of universality, namely the existence of definite phase transitions in the phase diagram of that particular system which we might call gravitation or spacetime.

One caveat needs to be spelled out immediately: theoretical physicists are well used to the fact that there are infinitely many different theories which might be taken as candidates to describe a single phenomenon, and that we need experiments to choose one of them, or more realistically to rule out as many theories as possible. So it should to be expected that there exist many different scenarios which could be called asymptotically safe quantum gravity according to the definition I provided above. In other words, there likely are many different phase diagrams corresponding to different systems, which we might all call gravitation or spacetime. Experiments will then be needed to rule as many asymptotically safe theories as possible.

The rest of this section is meant as an explanation or articulation of the tentative definition I have provided for the asymptotic safety scenario. This requires a short overview of concepts such as effective field theories, renormalizability, universality. After this introductory part aimed at conveying the fundamental concepts and ideas behind this approach to quantum gravity (QG), Sect. 9.2 briefly outlines some aspects of the progress accumulated over the decades in this field of research, with particular emphasis on the results concerning the quantum nature of

spacetime. Section 9.3 then addresses the fundamental and methodological role of the problem of time in the most commonly adopted theoretical framework for the investigation of this scenario. Finally Sect. 9.4 is devoted to the related problem of the cosmological evolution and of the origin of scales.

9.1.1 Effective Theories and Renormalizability

I would like to start from the notion of predictive power of a scientific theory. I believe most readers are familiar with this notion as well as with its failure. As an example, we know that classical electromagnetism is a highly predictive theory, since it is able to predict the outcome of infinitely many experiments by means of the four Maxwell equations and of the two natural constants contained therein (the electric permittivity and the magnetic permeability). But we also know that there are limits to the validity or utility of classical electrodynamics, because for instance it is not able to explain the stability of the atoms. In general, given a certain physical theory, it might be hard to forecast which are its limits of validity i.e., to which extent the theory can be expected to be predictive. In fact, in common jargon we tend to think that the harder is this guess, the more fundamental the theory is likely to be. As we will see shortly this jargon is reflected in the more specialized scientific language.

Quantum and statistical field theories are very special in this respect. They can say something about their own predictive power, which other theories cannot do. Field theories have this additional power, namely the ability to associate a characteristic scale to their predictions, and therefore to tell up to which scale they are predictive. The reason why this is so is simple: it is because they are theories describing an interacting continuous system, in which the dynamical laws not only regulate the behavior of the system at some length scale, but also prescribe how the

properties which the system exhibits at a first length scale are related to the properties visible at a second length scale. Again, this needs to be so by definition of fields permeating a continuous space. Think about a big elastic carpet like a trampoline: if you locally stretch it by stepping on it at one point, the whole configuration and tension of the carpet will change, at all points. If your weight were localized on an exceedingly small distance (as if you were wearing high-heels shoes rather than jogging shoes) the carpet would break, and at all points its description in terms of an elastic continuous medium would not be appropriate any more. So the local external pressure acts as a relevant scale in this system, because depending on its value the system shows two different phases, one with elastic properties and one without them.

Field theories which have a built-in maximum scale of validity are called *effective* field theories, a denomination which puts emphasis on their good property of being useful and predictive below this maximum scale. In fact effective field theories are so appreciated by contemporary theoretical physicists that they have become a golden standard in their methodology and vocabulary. Of course there are endless beautiful lectures about this subject, out of which we select few reviews for the interested reader (Georgi, 1993; Manohar, 2020; Bain, 2013). The scale that sets the boundary of the domain of validity of an effective theory is often called the *cutoff* of the theory. There are at least two sorts of such cutoffs. When the theory holds for lengths bigger (energies smaller) than some characteristic length, we talk about a ultraviolet (UV) cutoff; in the opposite case of a maximum length (minimum energy) scale the cutoff is said to be infrared (IR).

In contrast with effective theories are the so-called *fundamental* theories, where there is no limitation whatsoever, at least from the standpoint of pure theoretical

consistency, in the scale of applicability and predictivity. Having this definition at hand, we can now define the concept of a renormalizable theory. There are two main different meanings of renormalizability. A field theory is:

- *perturbatively renormalizable*, when, within a (all-orders) perturbative series, it appears to be a predictive theory up to arbitrarily small length scales, so no intrinsic UV cutoff is needed.
- *non-perturbatively renormalizable*, when, according to any method which goes beyond the technical limitations of perturbative methods, it appears to be a predictive theory up to arbitrarily small length scales, so no intrinsic UV cutoff is needed.

As we see, the two adjectives 'fundamental' and 'renormalizable' are affine, even if the latter is used with additional technical specifications which restrict its use to the scientific language. Now, strangely enough, the two definitions of renormalizable theory given above can be in contradiction with each other in all possible combinations. For instance, the theory of a real scalar field similar to a Higgs theory (ϕ^4 theory) in four-dimensional spacetime is perturbatively renormalizable but not non-perturbatively renormalizable. Also, the very same theory in a three-dimensional spacetime is not perturbatively renormalizable, but it is indeed non-perturbatively renormalizable.

Which of the two definitions is more desirable? Depending on the physicists you are asking to, you might get different answers. Asymptotically safe theories are a specific class of non-perturbatively renormalizable theories, and the definition of this class still requires the introduction of some more technical concepts. As you see, an important role is played by theoretical methods for performing computations, and providing a definition which encompasses all possible methods is a hard task. However,

there is some general non-perturbative understanding of quantum field theories which can be used to formulate more precisely the problem of non-perturbative renormalizability, in a way which almost method-neutral, in the sense that it can be implemented in almost all available methods. This is the topic of Wilson's understanding of RG flows.

9.1.2 The Renormalization Group Flow

In this section we aim at providing an operational definition of non-perturbatively renormalizable theories, which was invented by Wilson in the early 70s (Wilson & Kogut, 1974). Wilson's work is much more than that, because as it often happens, in trying to solve a problem it created a set of tools which survived the problem itself, and became a standard in scientific thinking. The main tool is the understanding of the so-called renormalization group (which had been discovered in quantum field theories starting from the mid 50s by a series of authors (Stueckelberg & Petermann, 1951; Gell-Mann & Low 1954; Bogolyubov & Shirkov 1956) in terms of a series of effective theories, and the rephrasing of renormalizability as an analogue of a critical phenomenon, i.e., a phase transition, in a statistical system. In many respects, Wilson's work has been inspired by, if not a field theoretical rephrasing of, the mid-to-late 60s works by Kadanoff on phase transitions (Kadanoff, 1966).

The simplest way of addressing these ideas is indeed in terms of Kadanoff's coarse-graining transformations, originally called spin-blocking because intended to take place in the Ising model, a lattice of spins interacting only with their nearest neighbors. Consider a discrete system such as a set of spins located on the vertices of a spatial lattice, with characteristic lattice spacing a, see the left panel of Fig. 9.1. Let's give the name S_a to a theory which

describes the interaction among the spins on the lattice with spacing a. It is possible to construct a mapping, that is a mathematical transformation rule called spin blocking or more generally coarse graining, from one spin system to a seemingly different one, living on a spatial lattice with different spacing a' and different interactions encoded in the second theory $S_{a'}$, with the property that all statistical/thermodynamical observables remain unaltered. Actually, there are infinitely many such mappings. See Fig. 9.1 for an example.

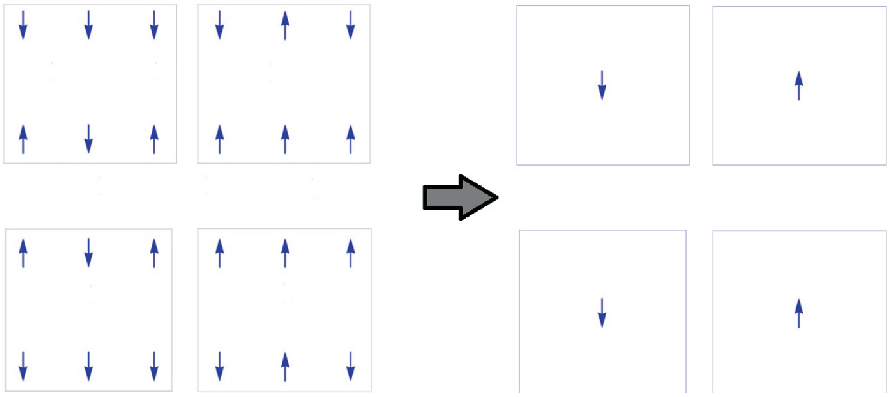

Fig. 9.1 A representation of a spin-blocking transformation which follows the majority rule. A two-dimensional lattice of spacing $a \times a$ is mapped onto another lattice of spacing $3a \times 2a$, by replacing each block of 3×2 sites and spins with a single site and spin. The new lattice site inherits the spin orientation which occurs most times in the original block of spins

Once a mapping from one lattice of spins to another is prescribed, this automatically induces a mapping between actions, with the property that statistical averages remain unaltered. The intuition by Wilson is then, that as long as the lattice spacing is finite, there is no fundamental difference between such discrete statistical models and a quantum or statistical field theory with a UV cutoff: in both cases there exists a smallest length scale, and therefore the physics is not that of a continuous system. This coarse-graining mapping is possible also among effective field theories with different UV cutoffs. In fact, in a field theory set up it is possible to consider infinitesimal changes of the

cutoff, thus building a continuous mapping in the space of theories (i.e., in the space of actions), which is called the renormalization group (RG) flow.

But what is the utility and the physical content of this RG flow? We argued that observable quantities are always left unaltered along the RG flow, hence this transformation seems to be completely un-physical and uninteresting, in the sense that whatever changes must be un-observable and un-physical. It is indeed so in many respects, apart for one exception, which justifies the whole game. Suppose the RG flow admits a fixed point, that is a special theory which is mapped onto itself. For such a fixed-point theory, nothing changes along the flow: changing the cutoff scale leaves the theory unaltered, which means that this special symmetry enjoys a scale symmetry, and describes a system with fractal properties. As the RG flow leaves the whole theory unaltered, nothing in the theory can be labeled as un-physical. In principle everything in this theory is observable.[1] Clearly, the interesting part of the story happens at these fixed points. From the point of view of the old problem of renormalizability, these fixed-point theories are a natural embodiment of the idea of non-perturbatively renormalizable theories. Indeed, as the RG flow changes the UV cutoff of the theory, and as nothing happens to the fixed-point actions, one can change at will the value of the cutoff at the fixed point, and also completely remove it (the minimum length to zero, or the maximum energy to infinity). Hence, fixed-point theories are predictive at all energy scales. But what about non-fixed-point theories? Are they renormalizable or not?

The RG is useful also to answer this question. Although we argued that for non-fixed point theories the flow itself is un-physical, it is a mathematical fact that at least some properties of the flow, in vicinity (in the infinitesimal neighborhood) of the fixed point, are as fixed and physical

as the fixed point itself, and are the same for any possible choice of RG transformation. This is just a consequence of the structure of the equations describing the RG flow, and is in perfect analogy with well known properties of classical-dynamical systems, where time evolution is the formal analog of the RG flow. Among these properties are the so called critical exponents. One can define one such exponent for each independent direction in theory space around the fixed point. The sign of the exponent determines whether the flow is attracted or repelled by the fixed point along the corresponding direction. Thus one can classify all possible terms in the action according to their behavior in vicinity of the fixed point. If we choose the convention of directing the arrow of the RG trajectories towards the IR (bigger lengths, lower energies) then:

- *relevant* operators, are those interaction terms which are repelled by the fixed point;
- *irrelevant* operators, are those interaction terms which are attracted by the fixed point.

The terminology is justified by the fact that in the particular case of the Gaussian fixed point, which is the point of vanishing interactions, the relevant/irrelevant interactions become stronger/weaker along the flow respectively. In statistical field theory, that is in the theory of phase transitions, relevant operators correspond to the thermodynamic variables which can be tuned to move across a phase transition (such as for instance temperature for the ferromagnetic transition, well described by the Ising model). Since the RG flow equations have the structure of non-singular partial differential equations, by continuity these directional properties of the flow are valid at least in a certain finite open neighborhood of the fixed point. Then, we can ideally follow the RG flow of any arbitrary non-fixed-point theory down towards the IR. The components of this arbitrary theory along the irrelevant directions will get

attracted towards a certain hypersurface, whose dimension is the same as the number of relevant operators. In fact, this hypersurface, called the UV-critical hypersurface, can be defined as the locus of points whose RG flow, when reversed in direction towards the UV, ends in the fixed point. The relevant operators can be used as coordinates of this hypersurface. Thus, any arbitrary theory which passes through a neighborhood of the fixed point will be quickly attracted to this special hypersurface (see Fig. 9.2 for a sketch of this situation).

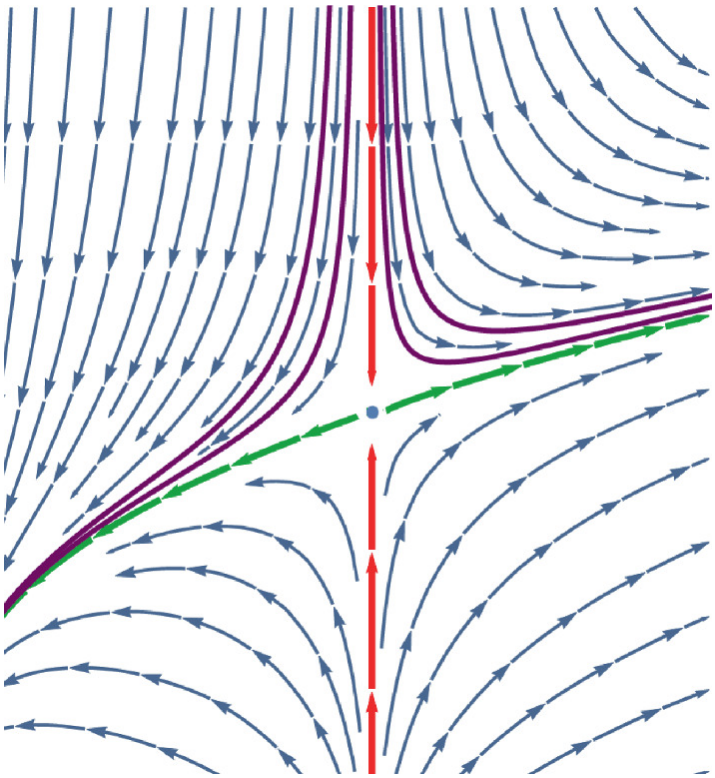

Fig. 9.2 A toy-model example for an RG flow in a two-dimensional theory space, from (Eichhorn, 2019). Here a fixed point represented as a blue dot at the center of the picture has one UV-attractive and one UV-repulsive direction. The UV-critical surface is indicated in green, the IR-critical surface in red. The RG flow is universally attracted towards the UV-critical surface in the IR, such that its characteristic properties are approximately realized also for theories (as the purple trajectories) that start off such surface

Then, for the task of computing the properties of this theory at large enough length scales, we can safely assume such properties will be the same as those of the theories

lying on the UV-critical hypersurface. In other words, if we know what this hypersurface is, the results of measurements taking place at large enough energy scales will be predictable in terms of a number of relevant parameters. Hence, there really isn't much of a difference among theories away from the fixed point. It is the mere existence of a fixed point, together with the possibility to compute its spectrum of relevant parameters, which allow us to conclude that any of the quantum field theories corresponding to RG trajectories passing through a neighborhood of the fixed point is a predictive theory, as far as the large distance properties are concerned, provided the number of relevant parameter is finite and small. This realization leads to the introduction of a third definition of renormalizability. A family of field theories is called *renormalizable á la Wilson*, if in their RG flow a fixed point exists, with a small finite number of relevant directions. If a theory is renormalizable á la Wilson, then it is also non-perturbatively renormalizable.[2]

A subtle but important twist in the meaning of the terms we have used so far occurs here. We had introduced the adjective fundamental for theories which are consistent and predictive at any energy scale. We also defined non-perturbatively renormalizable theories as those that are predictive at all energy scales. If we accept to call renormalizable á la Wilson a family of models, when a suitable fixed point exists in the RG flow of this family, we are tacitly accepting to be practical and limit our idea of predictivity to a certain range of energy scales or, that is the same, to a certain precision in measurements. In fact, *the whole family* of models, i.e., any RG trajectory, leads essentially to the same predictions as far as low-energy phenomena are concerned. Subtle deviations from theory to theory do occur at some high energy, or are visible at

any scale if high-enough precision is attained in experiments.

In the theory of phase transitions these deviations correspond to the so-called corrections to scaling. To assess whether nature is described by one particular representative in the family of theories, we would have to measure these deviations, which is a sub-leading task with respect to the primary task of finding a family of models which provides the right leading predictions for the visible effects at low energies. This pragmatic point of view is the substance of the Wilsonian approach. However, if one wants to be more rigorous, *only a sub-family* of these models corresponds to fundamental field theories, namely those models lying on the UV critical surface. In fact, only for them we can be sure that the UV cutoff can be removed, since the infinite energy limit is under the control of the fixed point.

The Wilsonian definition of renormalizability finally brings us to the key notion of asymptotic safety. It is customary language to distinguish two different kinds of Wilsonian renormalizable theories:

- *asymptotically free*, when the fixed point describes a non-interacting (also known as a free, or Gaussian) theory;
- *asymptotically safe*, when the fixed-point theory is instead interacting.

In the former case, the neighborhood of the fixed point can by definition be accurately described by perturbation theory, and this is therefore the most well understood and long studied case. The latter case has for a long time remained an exotic possibility, at least as far as four-dimensional-spacetime quantum field theories are concerned. In any case, regardless of the dimensionality of spacetime, non-perturbative methods are best suited for the study of this possibility.

Both scenarios have been proposed as possibly occurring in quantum gravity (QG). In 1977 Stelle discovered that a version of higher derivative QG, more specifically a theory containing only terms quadratic in curvature tensors, is perturbatively renormalizable and in fact asymptotically free (Stelle, 1977). Even earlier Stephen Weinberg, who had been one of the pioneers of the problem of perturbative renormalizability in quantum field theories and gravity, after studying the work of Wilson, in 1976 proposed that gravity might in fact be asymptotically safe (Weinberg, 1978), and in 1979 showed hints in this direction by discovering that standard Einsteinian QG has an interacting fixed point in $2+\epsilon$ dimensions (Weinberg, 1980).

This officially started the investigations of ASQG, which boomed only 20 years later. Before addressing a review of some of the progress made on this topic in the last 20 years, I would like however to dive into one last conceptual interlude, about what is possibly the most important advantage, or the deep fundamental meaning, of non-perturbative renormalizability.

9.1.3 Universality

The fundamental quality of the RG idea is the fact that it appears to be a powerful instrument for disentangling what, in a given theory, is a redundant or secondary accessory, and what is instead essential to produce some predictions. In the previous section we reviewed how the RG, together with the existence of a fixed point, allows to classify the space of theories into relevant and irrelevant terms. This is a first weak form of the principle that the RG can be used to simplify the theoretical description of phenomena, while keeping fixed the predictions for the result of measurements: relevant operators being the only ones that really matter for determining the latter.[3]

However, the above mentioned simplifying picture is tied to the RG flow of a certain given family of quantum or statistical field theories, but it does not justify the choice of such field theories in the first place. In other words, it does not answer the following question: given a certain set of long distance properties of the system, which kind of theories can account for them? Are there even theories which are not field theories and that would produce the same predictions? For instance hydrodynamical, or discretized, or string-theoretical models, or whatever else?

In general, there happens to exist full freedom in constructing alternative theoretical explanations for one and the same set of experimental results. But what is the theoretical mechanism that explains why two apparently at-odds models are describing the same physics? Again, Wilson's understanding of the RG provides a neat answer. In whatever theory you might come about with, it is possible to define RG transformations, changing the value of some length or energy scale. We have already met such transformations for instance in discrete spin models, and in continuous field theory. Now, one can ask the question whether this RG flow possesses fixed-points. If so, what are their characteristic properties, such as the set of relevant and irrelevant terms (and the corresponding critical exponents and so on)?

It is at this point that some magic occurs. It turns out that, while human imagination in creating theories is wild and un-classifiable, the set of RG fixed points possibly occurring in such theories is quite limited and surprisingly accessible to a systematic classification. In other words, it very often happens that different and independently constructed theories show RG fixed points which are identifiable, i.e., there is actually one fixed point, such that the physical predictions arising in the part of the phase diagram controlled by such fixed point are the same. Two famous examples of this phenomenon are:

- A quantum field theory of a real scalar field with a quartic self-interaction (in two, three and four dimensional spacetime) which is identifiable with the so-called Ising model, a theory of one-component spins living on a discrete spatial lattice and endowed with nearest-neighbor interactions.
- The quantum field theoretic description of strong interactions, quantum chromodynamics (QCD), which is identifiable with (several different versions of) a theory of Wilson lines and color charges on a discrete spatial lattice known as 'lattice QCD,' invented by Wilson himself as a computer-friendly alternative to QCD itself, which in some number of spatial dimensions or in certain regimes it is also identifiable with a string theory.

As different models are often invented to describe different physical systems, the other-side of the correspondence among theories, is a correspondence among different physical phenomena, which is the glorious summit of the theory of phase transitions, and which has been given the somewhat bombastic name of 'universality.' Here is what it means. Different physical systems show phase transitions, i.e., sudden drastic changes in their macroscopic properties, as one varies the control parameters in a laboratory. These phase transitions occur in a specific way: we can measure how a certain property such as pressure or magnetization or conductivity changes as we approach the phase transition point. This defines the so-called critical exponents.

Universality is the phenomenon according to which one physical systems shows, at some of its own phase-transition points, the same critical exponents as another system at one of its own phase-transition points. For instance, the first example of identification of theories presented above, that is often called the Ising universality class, in three dimension describes: the transition between ordered and disordered states of a ferromagnet at the Curie

temperature; critical opalescence, a phenomenon where the mixture of two substances like water and alcohol suddenly jump from being transparent to opaque and then transparent again; the continuous transition between liquid and vapour phases which takes place at a special value of temperature and pressure known as the critical point.

Let us then summarize the path we followed up to this point. Perturbative renormalizability was meant to ensure that quantum field theories, within the application of perturbative methods, be computable and predictive. Beyond perturbation theory, this requirement is subsided and replaced by the requirement of non-perturbative renormalizability. An operative way of fulfilling the latter requirement is in terms of Wilsonian renormalizability which, besides enabling us to perform computations and make unambiguous predictions, it also allows us to compare our predictions with other possible theoretical models, to come up with alternative but equivalent theories for the same physical phenomena, or even to look for different physical systems which can be considered as analogues of the system we started with, in that their critical properties belong to the same universality class of the model we started with.

9.2 Recent Progress in ASQG

The modern approach to ASQG is continuously connected to the quantum-field-theoretical perturbative approach dating back to Feynman, Wheeler and DeWitt. In 1972 the first complete one-loop computation (of the effective action) in QG was performed by ’t Hooft and Veltman (Hooft, 1974). Here for the first time the problem of the perturbative non-renormalizability of QG (coupled to a scalar field) was met. In the following years several authors (most notably Deser and van Nieuwenhuizen) performed beautiful and sometimes very complicated computations

extending this result to pure gravity as well as gravity coupled to Standard-Model-like fields, at one and two loops. The final tombstone on this research project came with a couple of works where the generic perturbative non-renormalizability of QG theories at two-loops was established (Goroff & Sagnotti, 1986; van de Ven, 1992).

Being the 70s the years of the great success of quantum field theories, under the implicitly assumed paradigms of perturbation theory and perturbative renormalizability, these failures cast the highest emphasis on the different status of QG among the other known fundamental interactions. After these works, the QG scientific community was shattered and drifted towards a wide pool of alternative (to quantum field theory) approaches: supergravity and string theory, non-commutative geometry, loop quantum gravity, causal dynamical triangulations.

If there is any substantial departure of the modern reasearch on ASQG from those classic works, I believe it is mainly a conceptual one, consisting in the shift from the perturbative to the Wilsonian language, as I explained in the previous sections. Yet, a major role in the progress that this line of research has undergone in the last decades has been played by technical issues regarding the theoretical tools used for computations.

The advances which triggered the quick formation, in the first decade of the 21st century, of a young specialized community of theoriests working on ASQG mainly consist in the collection of an impressive number of different computations and approximations which all observe the existence of an interacting RG fixed point for the standard field-theoretical quantized version of Einstein's general relativity. This fixed point has a small number of relevant directions (three to four in most investigations, or just two in uni-modular gravity) indicating a high predictive power for this quantum field theory. For a detailed account of the bibliography see for instance (Reuter & Saueressig, 2019).

It is also remarkable that a compatible fixed point is found with apparently different field theory formulations of general relativity (different by parameterizations of the dynamical variables and/or by the fundamental symmetries of the classical theory). It is even more remarkable that at the same time, progress in different approaches to QG has converged with the evolution of the ASQG scenario, in that also in these alternative approaches some evidence has appeared for the existence of an interacting fixed point of the RG. Such approaches are:

- Regge's formulation of discrete lattice QG (Hamber, 2015; Falls, 2015, 2016; Biemans et al., 2017);
- the causal-dynamical-triangulation formulation of discrete lattice QG (Ambjørn, 2019);
- the reformulation of several triangulation prescriptions in terms of tensorial and group-field-theory models (Ben Geloun et al., 2016, 2018; Ben Geloun & Koslowski, 2016; Eichhorn et al., 2020; Geloun et al., 2024).

Furthermore, the implementation of RG methods and the search for fixed points has started also in spin-foam models (Steinhaus, 2020; Asante et al., 2022).

9.2.1 Are There Predictions for Observable New Physics?

While a large amount of work has been devoted to new physics stemming out of the ASQG scenario, as far as the observable/measurable phenomena are concerned, QG meets general strong limitations. These limitations are not special to ASQG, but rather stem from the requirement that the general relativity principle holds: predictions for result of measurements should be formulated in a way which does not privilege any possible observers in the universe over any other. This leads either to extremely cumbersome observables, such as the volume of the universe, or to relational quantities, such as the length of geodesics (i.e.,

the free-falling path separating two observables). Both kind of observables have been addressed in ASQG (Becker & Pagani, 2019; Becker et al., 2020; Baldazzi et al., 2022), an analysis whose conclusion is one more substantiation about the fractal structure of spacetime in ASQG, which we address in one of the following questions.

There are at least two alternative kinds of observables which one might address in QG. One is based on the assumption that a given background is physical in certain kinds of measurements, and therefore one can refer observables to it. For instance, this is the case of scattering (collisions) experiments like those studied with the LHC. One can then predict characteristic signatures of ASQG in the results of hypothetical such experiments able to probe the impossibly high energies typical of QG effects.[4] A more interesting kind of observable signatures, is the so-called indirect effects. With this, we mean observable phenomena which are not directly related to gravitation, but rather to other physical interactions such as the sub-nuclear forces and the particle-physics phenomena. As QG is expected to affect these interactions, one can elaborate theoretical scenarios for the interplay between these two realms. Prescribing that QG is asymptotically safe in these scenarios is typically very constraining. In other words, not any particle physics phenomena would be compatible with ASQG. Measuring the properties of these sub-nuclear interactions at the energies accessible to nowadays experimental setups is therefore indirectly testing this scenario. As an example of this mechanism, ASQG has been argued to favour a light Higgs mass of around 126 GeV, even before the value 125 GeV was measured by experiments (Shaposhnikov & Wetterich, 2010).

9.2.2 Which Is the Quantum Structure of Spacetime?

Spacetime as used in general relativity, for instance a metric tensor defining time and space intervals between events, is not an observable concept in QG, as discussed in the previous answer. Hence there is infinite freedom of changing the answer to this question, while keeping physical predictions unchanged. However, there are some universal features that must be reproduced in any of these possible answers, as they are directly related to observables. Most notably, ASQG predicts that the dimensionality of spacetime, whether defined by the scaling of volumes and geodesic lengths, or by the fractal properties of a RG running metric (Lauscher & Reuter, 2005), is different from four at small length scales. In particular, at high energies one observes a dynamical dimensional reduction to approximately two dimensions. Remarkably, this phenomenon has been observed also by a number of other QG approaches: causal dynamical triangulations (Ambjørn et al., 2005) and loop quantum gravity (Modesto, 2009, Magliaro et al., 2009) among them.

9.2.3 Is There a Minimal Length or Time Interval?

Again, this question is ill-posed unless one carefully defines an observable quantity which embodies the idea of length or time interval. In constructing such an observable, one faces a fundamental ambiguity: only dimensionless quantities, i.e., pure numbers free from unit of measures, are actually measurable. This means that the answer to the question depends on the units of measure one chooses, because different units have different RG flows and behave differently in the high-energy limit. Indeed, in one particular unit, such as Planck units (where one measures quantities as multiples of the Planck mass), a minimal length might exist, while in different units, such as electroweak units (where for instance one measures

quantities as multiples of the Higgs mass) there would be no such minimal length (Percacci, 2007, Percacci & Vacca 2010).

9.2.4 Is Time Emergent or Fundamental?

Before answering this question, it is better to try to avoid confusion in the use of the word 'time.' Hence, let us distinguish between two different notions of time.

- *Macroscopic time(s)*: the variable(s) which are as close as possible to low-energy observables in QG and in the classical regime of gravity. Examples for such observables can be found in cosmological, thermodynamic or anthropic contexts. As such, macroscopic time(s) must be present in any theory of QG, through the definition of observables (e.g. relational time intervals like cosmological red shifts).
- *Microscopic time(s)*: some purely mathematical entity, a label or variable, which might be present or not in a theory of QG. If present, it might well be that this microscopic mathematical entity has a different interpretation and different properties from the macroscopic time we defined above. However, the use of the word 'time' for this mathematical entity must be justified by the existence of a theory-specific relation, or mapping, between this time and the macroscopic or classical time.

Based on these definitions, I would propose the use of the following jargon. In a given theory:

- time is *fundamental*, if the theory features a microscopic time which appears as a necessary mathematical tool, a sort of basic building block for the theory which is present at all length scales, especially at the smallest scales;

- time is instead *emergent*, if a microscopic time never appears in the theory at small scales, or if its appearance is not ubiquitous and unavoidable, but it is rather a special and accessory appearance subordinated to the focus on some very particular phenomenon. We then use the adjective emergent, as the macroscopic time of experiments will have to be accounted for by the theory even if the basic variables one uses to make predictions do not include a microscopic time variable.

Incidentally, the previous definitions and jargon could be generalized from time to any other labeling, such as mass, charge, temperature and so forth. Notice a strict similarity and parallelism between these two definitions and the ones of fundamental and effective theories given in Sect. 9.1.1. With these definitions I have likely oversimplified the problem, by insisting in a sharp distinction between 'microscopic' time as a prime actor on the stage of the theoretical description, and 'macroscopic' time as a passive spectator in the audience of predictable measures. Yet I believe that in many cases this distinction can be made. For instance, in the conventional description of ASQG in terms of a quantum field theory of metric or connection degrees of freedom, defined on a spacetime, there definitely exist a microscopic time. The same can be stated for string theory, or for non-commutative geometry. On the other hand, according to the experts, there is no microscopic time in loop QG, nor in tensorial/group-field theory approaches, nor in energetic causal sets. Can we then conclude that in ASQG time is fundamental? Actually no. The correct conclusion is that in one theoretical formulation of ASQG time is fundamental. But in another formulation time might be emergent. The crucial point is that the physics of ASQG is not the same as its theoretical formulation. As we argued in Sect. 9.1.3, the requirement of asymptotic safety, which defines ASQG, is in fact enough to guarantee that there will

be infinitely many theoretical descriptions of one and the same physical system. ASQG is a name for the physical system and its properties, not for its particular description in terms of a quantum field theory of the spacetime metric. Indeed, as we argued in the previous answers, it might well be that in the next decades some alternative but equivalent description of ASQG in terms of a loop QG or of a causal dynamical triangulation would emerge. Hence, using the above-defined jargon and distinguishing between emergent and fundamental time, while possible at the theoretical or conceptual level, does not directly address the physical phenomena which are experimentally tested. In this sense, the distinction is unphysical and misleading.

With this caveat clearly spelled out, the rest of this article is devoted to a more specialized discussion of some aspects of the problem of time, in one particular family of theoretical representations of ASQG, the one that for historical reasons is most associated to this name. This is the formulation in terms of a relativistic quantum field theory including a spacetime metric and/or connection as a dynamical and quantum degree of freedom.

9.3 The Methodological Problem of Time in Asymptotically Safe Quantum Gravity

9.3.1 The General Relativity Principle: Gauge Invariance

General relativity is a gauge theory, meaning that it possesses a local symmetry: the freedom to change spacetime coordinates at any point in space and at any instant in time. The fact that this transformation leaves the physics unaltered is the so-called principle of general relativity or covariance. Respecting this kind of symmetries

while quantizing a field theory is especially challenging from the technical point of view of the mathematical formalism of field theories. This is the case in any gauge theory, for instance also in electrodynamics and in the standard model of sub-nuclear interactions. Gravity is an especially difficult case because the symmetry group of general coordinate transformations is a highly nontrivial one.

Thus, the fundamental principle of general relativity is the starting point of one important aspect of the so called “problem of time” in quantum gravity. Because time evolution is tantamount to a reparametrization of our time coordinate, i.e., it can be interpreted as a continuous re-tuning of clocks, it is a symmetry transformation itself, and observable quantities must be insensitive to it. From the point of view of the so-called canonical quantization prescription (one of the many equivalent mathematical formalisms for constructing a quantum theory out of a classical one) this appears paradoxical, as the Hamilton function (a mathematical embodiment of the total energy of a system), being the generator of time evolution, must vanish, i.e., it must leave all physical states unaltered.

From the point of view of other quantization prescriptions, such as for instance Feynman’s functional integral quantization, this might not appear paradoxical at first, but still represents an undeniable challenge. In any QFT representation, this gauge symmetry of Einstein’s theory of gravity (and often also of alternative theories of gravity) entails that the quantum theory must fulfill some constraints or identities, which are typically very nonlinear and complex to solve. These constraints take different names depending on the particular mathematical formulation adopted: Ward-Takahashi identities, Slavnov-Taylor identities, master equation etc.

In the field theoretical framework of ASQG, this technical challenge is approached following the same

strategies at work in quantum electrodynamics or in the standard model, which are various and diverse, but can be classified in two distinct families.

- *Symmetry breaking* strategies: in which the constraint identities are *not* solved at each step of the computations, but rather as a last moment of the theoretical treatment. In other words, one first works with a theory that does not possess this symmetry, and at last tries to extract from it predictions that would apply also in the symmetric theory.
- *Symmetry preserving* strategies: in which the task of fulfilling the constraints is taken as primary, and the field theoretical tools are specifically chosen to ease this process, often at the expense of loosing generality and possibly making other kinds of computational tasks harder.

Most of the literature on ASQG follows a mixed route, in that it takes care of preserving only part of the symmetry, while breaking the rest (and sometimes trying to recover it in the end). This happens through the use of a formalism called background field method. While this is not a mandatory choice, understanding the status-quo of the gauge problem in ASQG requires an in-depth discussion of the background field method.

9.3.2 Background Spacetime and Invariance Therefrom

Despite the background field formalism is especially useful in perturbative computations of gauge theories, it has been adopted in the nonperturbative investigations of ASQG because the latter make crucial use of the Wilsonian RG. As we discussed in Sect. 9.1.2 the definition of the RG flow involves a coarse-graining process, which corresponds to reducing the spatial resolution (i.e., removing the quantum

fluctuations with high energy), such as the lattice spacing of Fig. 9.1. Most of the RG mappings, or RG schemes as they are also called, therefore make explicit reference to a unit of length or mass/energy. Such RGs are therefore named mass-dependent schemes. There are exceptions, called mass-independent schemes, which we will discuss briefly at the end of this section. Because the spacetime metric is by definition the mathematical entity that measures lengths and energies, the construction of a mass-dependent RG scheme must involve a spacetime metric. It is in principle possible to use the full quantum metric of spacetime to define the RG. This is often done in the symmetry preserving strategies (Wetterich, 2018; Becker & Reuter, 2020, 2021), such as for instance in higher derivative regularization schemes (Slavnov, 1972; Lee & Zinn-Justin, 1972; Falls, 2021). This choice is however particularly unusual and inevitably complex, because in QG the spacetime metric is also the quantum field to be averaged during the coarse-graining process, i.e., the analog of the spins of Fig. 9.1. In other words, and by slight abuse of language, in QG the microscope setting the resolution with which we can observe the system, and the system we want to observe, are one and the same thing.

In normal field theories, unlike in QG, the coarse graining can be defined without involving the quantum field itself, and this makes the RG flow much simpler for the sake of both computations and interpretations. The background field method offers us a chance of applying this kind of understanding of the RG also to the QG case. It consists of making use of the ubiquitous freedom one has in a field theory of redefining the fields, and to split the spacetime metric into two parts: a background metric which is treated as a classical or external or reference field, and a quantum metric which accounts for the fluctuations around the former.

Despite the terminology might be misleading, the use of a background spacetime as an auxiliary tool for computations does not necessarily imply that the predictions the theory makes do depend on the choice of a particular background. On the contrary, independence from the background is still attainable under some specific circumstances. The first of these circumstances is the easiest property to be fulfilled and guaranteed: independence from the choice of coordinates on the background spacetime. In fact, because general changes of coordinates are gauge transformations of the spacetime metric, in the background field formalism one can define two classes of gauge transformations, roughly speaking: gauge transformations either of the background or of the quantum field. While the task of maintaining the quantum gauge symmetry is as challenging as it was without the introduction of a background field, the preservation of the background gauge symmetry is much simpler, because the latter is a classical symmetry.

Thus, while essentially all ASQG studies automatically preserve background gauge symmetry, many of them ignore the constraints stemming from the quantum gauge symmetry. Still, what one aims to is extracting results which would be valid in case this symmetry was present. The way in which this happens is generally different from study to study. I will limit this discussion to presenting one popular argument, which is often adopted also outside of QG, for instance in QCD (Ellwanger et al., 1996).

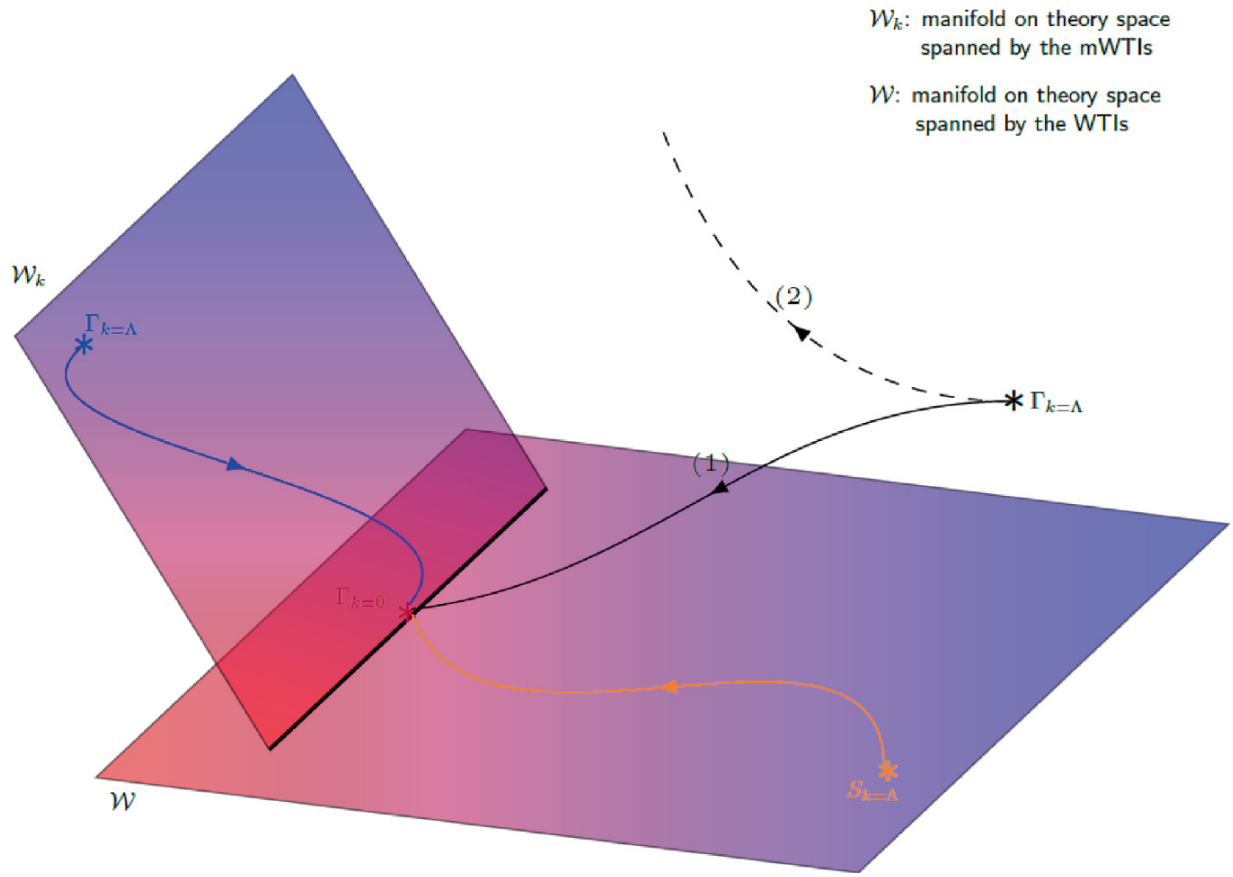

Fig. 9.3 A pictorial representation of soft-breaking vs hard-breaking vs preservation of a symmetry, from Gkiatas (2023). Preserving a classical or quantum symmetry requires solving some complex Ward-Takahashi indentities (WTIs) thus remaining on a sub-space of theories. Working off these spaces corresponds to breaking the symmetries, either softly, as in case of trajectory (1) when the RG flow leads to the symmetric situation at large distances, or hardly as in case of trajectory (2)

The argument relies on the Wilsonian classification of interactions as relevant or irrelevant. Ideally, not taking care of preserving some symmetry constraints corresponds to taking into account a more general class of field theories. In fact, a set of constraints can be interpreted as defining a sub-family of theories or, in geometrical language, a sub-space in the functional space of theories. In Fig. 9.3 two examples of such subspaces are pictorially represented: $\mathcal{W}$ represents the subspace of theories which fulfill a purely classical symmetry (encoded in some Ward-Takahashi identities), and $\mathcal{W}_k$ represents the subspace of theories enjoying a quantized version of the same

symmetry (encoded in some modified Ward-Takahashi identitites). If the computations performed in the more general space of theories show the existence of RG fixed points, and allow for the classification of interactions into relevant and irrelevant around each fixed point, then through this understanding we can in principle classify the effect of those interactions that break the symmetry as either relevant or irrelevant (in vicinity of the fixed points). Once this is achieved, in order to extract predictions valid for the symmetric theory all one needs to take care of is setting to zero the value of the relevant symmetry-breaking operators (if any), since the irrelevant ones will not affect the main physical predictions of the theory, due to the Wilsonian arguments presented in Sect. 9.1.2. Since typically the RG fixed points tend to show only a finite and small number of relevant operators, setting to zero a subset of them is a much simpler task than demanding *all* symmetry breaking operators to vanish, the latter typically being an infinite family, whose precise definition is tantamount to solving the difficult symmetry constraints. This strategy is called soft-breaking of a symmetry. In Fig. 9.3 a situation of soft-breaking corresponds to the black solid RG trajectory in theory space, where at large distances the flow leads to the same theory as in the classical or quantum symmetric case. A situation where some symmetry-breaking relevant operator is not removed leads instead to the black dashed trajectory, whose low-energy predictions substantially differ from the symmetric theory.

This soft-breaking mechanism lends us an interpretational key which can be applied in a quite general class of approaches, sometimes even outside the first family of symmetry-breaking strategies. In fact, a certain family of symmetry-preserving strategies is based on the above mentioned massless RG schemes. The latter can be

interpreted as stemming from mass-dependent schemes through a limiting process. Typically one first starts in a situation where the theory possesses two independent RG scales, for instance an UV scale Λ, like the lattice spacing in a spin system, and an IR scale k, like the total number of lattice sites. Then one obtains a new single-scale RG scheme by removing one of the two scales, for instance sending the ratio $\Lambda/k \to 0$ at a fixed value of Λ (Bain, 2013; Baldazzi et al., 2021a; Branchina et al., 2022). This limit can be used to simplify some computational task, again for instance because if taken appropriately it can automatically remove the symmetry-breaking irrelevant deformations as in Fig. 9.3. This mechanism is working fine in perturbation theory, where a massless RG scheme like dimensional regularization in $\overline{\text{MS}}$ has become the theoretical standard of approaching gauge theories. Applications beyond perturbation theory and in ASQG have started but are still in an exploratory phase (Baldazzi et al. 2021a, b; de Brito & Eichhorn, 2023). Other symmetry-preserving schemes have been developed or proposed over the course of the last few decades, such as the above mentioned higher-derivative regularization schemes, propertime regularization (de Alwis, 2018; Bonanno et al., 2020), supersymmetry-based Pauli-Villars regularization (Morris, 1998; Arnone et al., 2002, 2003, 2005, 2007; Morris & Rosten, 2006), BRST-based regularization (Asnafi et al., 2019; Gies et al., 2022; Gkiatas, 2023). Yet, none of them has managed to become more practical and popular than the symmetry-breaking strategies outlined above.

Besides the symmetry constraints, background independence requires also an additional circumstance, which in turn is related to one more constraint on the theory. This is the so called quantum equation of motion. The background metric is an external field introduced in the theory ad hoc, and from this point of view it is not very

different from just as another external-source or classical-probe term in the functional integral. As such, its introduction might or might not irreparably change the dynamics of the system. This in turn depends exclusively on the system itself and its stability properties, i.e., ultimately on its phase diagram. Just like for other source terms, we use arbitrary values at first as a tool to ease computations, but to learn about the system properties we finally need to fix the value of this external field by means of some associated constraint. In the case of the background field, this constraint is called background equation of motion. The latter stems from the usual classical stationarity principle, but it contains information about the full quantum dynamics: it is equivalent to the fact that the system prefers values of the background metric which minimize the quantum effective action. This is the last challenging functional constraint that needs to be solved to ensure full background independence in ASQG (Pagani & Reuter, 2019; Lippoldt, 2018).

9.3.3 Causality and Unitarity

It would be a negligence, in a chapter dedicated to the problem of time in ASQG, not mentioning the relevance of two key fundamental concepts such as causality and unitarity. Their relation with time in ordinary quantum field theories is clear:

- global causality, associated to a rigid Minkowski spacetime, is part of the principle of special relativity which a relativistic quantum theory must obey;
- unitarity in the space of quantum states, i.e., the conservation of norms along every energy-preserving flows, among which time evolution itself, is demanded by the necessary existence of a probabilistic interpretation of measurements according to the basic postulates of quantum mechanics.

Regarding these properties, QG is special in various respects. In QG causality appears to be promoted to a local property valid in every coordinate system by the general relativity principle. But even general relativity is known to possess some subtleties concerning global causality, in that this might be violated in specific spacetimes. For instance it is well-known that some solutions of the equations of motion, such as rotating black holes, appear to violate this principle 'spontaneously.' Whether this is the case also in QG or not, is an open problem in any QG formulation, and also in ASQG. For a discussion of this topic and how it could be attacked from the ASQG point of view see Platania (2022).

Unitarity also appears to be generalizable to the case of QG, although in formulating it precisely one soon hits the problem of constructing a Hilbert space of states, which requires solving the previously-mentioned problem of time of QG. Fulfilling the gauge-symmetry constraints as well as all the quantum equations of motions is a prerequisite for addressing this issue. This is a general feature of any gauge field theory. If the gauge symmetry is broken, or if the theory still contains external arbitrary classical fields, the space of states is inevitably deformed and it contains also states which would not be present in the genuine symmetric system in vacuum (Henneaux & Teitelboim, 1994). Furthermore, discussing the unitarity of a QG theory also needs as a prerequisite a complete solution to the question of background independence. If the background spacetime is not dynamically determined by QG itself as a solution to the quantum equations of motion, but is rather fixed as a book-keeping tool, no fully consistent answer to to the unitarity question can be provided. In fact, the lesson that quantum field theory on curved backgrounds and semi-classical QG teach us, is that the construction of a unitary space of positive-energy states or equivalently of a physical scattering matrix might be impossible on some background

spacetimes, which in turn might either signal the problems of the interacting theory itself at the quantum level, or rather the instability of the chosen background.

Thus, addressing these two properties is a formidable task and indeed only recently some partial and exploratory results have been obtained in ASQG. Some studies have relied on dimensional reduction or the analogy with two-dimensional conformal theories (Nink & Reuter, 2016). Other studies have accepted to compromise and start investigating these issues around a specific background, such as a Minkowski spacetime. This can be done either for un-observable quantities like the graviton propagator, the spectral density functional or form factors (Bonanno et al., 2022; Fehre et al., 2023; Knorr et al., 2022), or for observable quantities like scattering amplitudes (Buccio et al., 2024). While promising, these directions of research are still at the beginning as far as the ASQG scenario is concerned.

9.4 The Cosmological Problem of Time: The Origin of Scales

The aspects of the problem of time that have been discussed so far are rather abstract and theoretical, and might appear far from observable natural phenomena. In this section I would like to highlight their connection with a more empirical manifestation of the problem of time: the cosmological arrow of time. The cosmological evolution poses us many questions and is an extremely rich field of research. One question is however of primary and fundamental relevance: how did this evolution originate? Why do we observe time rather than timelessness?

This question is closely connected with the concepts we reviewed in Sect. 9.1. Understanding how this is so only requires a simple rephrasing. The cosmological history is

an evolution in the characteristic energy scales of the physical phenomena taking place in the different ages, or equivalently in the typical length scales of the structures generated in each age. In other words, time, energy, distance are different manifestations of one unique aspect: the existence of scales, which in the language of theoretical physics is called scale-symmetry breaking.

A physical system is scale symmetric if the results of measurements are independent from the typical spatial-temporal resolution of the observation. Since the result of an experiment is always a dimensionless number, this means that the dimensionful theoretical observable quantities always scale according the right power of the corresponding units of measure, to ensure that the dimensionless quantity is independent from the actual choice of such units. This is precisely the scaling property postulated by Kadanoff at a continuous phase transition, which occurs at a fixed point of the RG flow (Kadanoff, 1966). Clearly, absence of scale symmetry corresponds to being away from an RG fixed point. So, a theoretical model that aims at explaining how scales and their evolution come into being, necessarily must be able to explain how the RG evolution of the theory ultimately connects with some fixed-point behavior.

ASQG is in a mildly favourable situation in this respect, as it makes reference to the RG properties of QG as foundational and defining for the asymptotic safety scenario. By requiring the existence of a suitable RG fixed point which can be connected to a long-distance quasi-classical regime by an RG flow towards the IR, ASQG possesses all the theoretical instruments needed to study the generation of scales. This circumstance applies to any scenario of Wilsonian renormalizability.

How is it possible to describe the generation of scales in a Wilsonian language? There are two main theoretical

mechanisms for the breaking of scale symmetry in a quantum field theory:

- *dynamical symmetry breaking*, also called (especially in a perturbative setting) *radiative symmetry breaking*, when the deviation from the fixed point that leads to scale generation corresponds to a relevant operator;
- *spontaneous symmetry breaking* when the theoretical laws themselves preserve scale symmetry, i.e., they do correspond to an RG fixed point, but the quantum states do not, because the symmetry-preserving states happen to be metastable to external perturbations.

A note of caution is appropriate here, since in most quantum-field-theory literature the name dynamical or radiative symmetry breaking is reserved to a special category of relevant operators, the so-called marginally relevant operators, which are characterized by small critical exponents. In the particular case of the Gaussian fixed point, marginal operators correspond to those coupling constants which would be dimensionless according to classical scaling (naive dimensional analysis), but acquire some small nonvanishing anomalous dimensions due to quantum effects. An example is the fine-structure constant in electrodynamics. These are contrasted with relevant interactions, whose couplings would have nontrivial dimensionality already in the classical theory. An example is a mass parameter, or the four-fermions contact interaction strength of Fermi's theory of weak nuclear forces. Such parameters, when non-vanishing, would break the scale invariance of the theory already at the classical level. Therefore in a perturbative context the breaking associated with these relevant interactions is called *explicit symmetry breaking*. Out of the perturbative context, since the value of critical exponents at a nontrivial fixed point are *a priori* unknown, it does not make much sense to qualitatively differentiate between

large and small critical exponents, i.e., between relevant and marginally relevant interactions. In practice, at the quantitative level, it does of course make an important difference.

Each of the above-mentioned scenarios in principle leads to predictive signatures that can be used to confirm or rule out the precise mechanism of scale generation. However, as is often the case, extracting these predictive signatures is a complex computational task. This task is already hard in perturbation theory where the fixed point controlling the system is the well-known Gaussian one. In fact, even in this case the scale generation might either occur in the weakly coupled regime, such as for instance in the usual treatment of radiative symmetry breaking in the Higgs sector of the Standard Model, or in the strongly coupled regime, such as for instance in QCD according to the standard understanding of confinement. One should stress that already in gravity-free theories characterizing and differentiating these two scenarios requires a full understanding not only of the UV fixed-point regime but also of the IR behavior, i.e., of the mapping between the microscopic UV description and the particle spectrum visible at low energies. For simple Yang-Mills theories this is a one-million-dollar problem.

In theories accounting only for gravity (pure QG), the first of these scenarios have been investigated especially for cosmological predictions (Bonanno & Reuter, 2002a, b; Contillo et al., 2012; Bonanno, 2024), among which a special mention goes to purely-gravitational inflation (Weinberg, 2010; Bonanno et al., 2011; Contillo et al., 2012; Copeland et al., 2015; Bonanno & Platania, 2015). The second scenario in pure gravity has also received attention for its high cosmological predictivity (Wetterich, 2019, 2021, 2022; Shaposhnikov & Tokareva, 2023a). Beyond inflation and dark energy, cosmology is ultimately a complex subject and one needs to account for all the

fundamental interactions besides gravity. Generalization of these scenarios in a full-fledged beyond-the-Standard-Model theory that includes an ASQG sector is not only technically hard, but it also poses conceptual questions. In a theory with many independent fields and parameters there exist many independent ways of breaking scale symmetry, which might occur separately or concurrently. For instance, in the Standard Model the Higgs sector and the QCD sector both contribute to the generation of particle masses. In particular, at energies of few MeV the strong nuclear forces between mesons and hadrons are the main actors, while the Higgs sector is completely decoupled because of the high masses of the Higgs and W bosons. Instead, at energies of the order of some TeV the scale-symmetry breaking due to QCD is assumed to be negligible due to the weak value of the strong coupling, while the Higgs contribution is usually assumed to dominate. But the explanation of how the Higgs sector breaks scale symmetry and becomes massive, the so-called electroweak symmetry breaking, is not offered by the Standard Model, and it might well be related to how the QCD sector by itself generates the masses of mesons and hadrons (Holthausen et al., 2013; Lohitsiri & Tong, 2019; Gies et al., 2023).

These questions become even more convoluted if one wants to additionally describe how the Planck mass, i.e., the characteristic energy scale of quantum gravitational effects, is generated out of a scale-invariant situation. This finally calls into the scene the so called gauge-hierarchy problem: why are there many orders of magnitude of difference between all these masses? Despite explaining the precise value of many parameters in a single theory is an ambitious goal, typically one expects that there should be a common origin for all these masses, because of a phenomenon called *vacuum alignment* in the mechanism of

symmetry breaking. Roughly speaking, this means that once a symmetry is broken by identifying a preferred direction, then the breaking of further symmetries will find a favorable ground along the already established special direction. So, the first breaking acts as a seed for a cascade of subsequent breakings. Based on this expectation, and because of the high predictivity of this scenario (low number of free parameters), a special interest has been attracted by theories where a single mechanism of scale symmetry breaking can be used to account for all masses and energy scales of interest. In other words, the Planck mass and the Higgs mass are expected to be originated in one shot from a single scale invariant situation (Shaposhnikov & Shkerin, 2018; Shaposhnikov & Tokareva, 2023b). In ASQG this kind of scenario is conceivable (Shaposhnikov & Wetterich, 2010), but requires the inclusion of matter fields and their interactions, which leads to a highly complex theory, whose exploration has been the subject of several decades of studies (Eichhorn & Schiffer, 2022). Interestingly, scholars have recently started computational analyses accounting for the full Standard Model augmented by ASQG (Pastor-Gutiérrez et al., 2023). This research direction is likely to represent one of the frontiers for ASQG in the next decades, and it will hopefully bring some contribution to advancing our understanding on the origin of scales and time.

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Footnotes

1 One should be careful here. As we said before, there isn't just one RG mapping but rather infinitely many. Hence, what stays fixed along one RG mapping could change along a different one. Thus, more precisely, the observables are those quantities which stay fixed along any possible RG transformation.

2 The vice versa might however not be true. In fact, Wilsonian renormalizability is a sufficient but not necessary condition for non-perturbative renormalizability.

3 Incidentally, this part of the Wilsonian understanding of field theories has been used to disregard its second part, concerning renormalizability, and to raise the effective-field-theory paradigm to the status of the all-encompassing standard of modern high-energy theory.

4 The typical QG energy scale is the infamous Planck scale of 10^{19} GeV, i.e., 10^{16} times higher than the typical energy scale probed at LHC.

Part III
Time and Timelessness in Cosmology

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10. Time, Spacetime and F-Theory

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Abstract
This Chapter explores the philosophical and ontological implications of F-theory, a non-perturbative extension of Type IIB string theory, mainly focusing on the apparent existence of two temporal dimensions. The paper proposes an interpretation that advocates for a single temporal dimension grounded in a brane-based ontology, challenging

the conventional understanding of time and spacetime in F-theory. Moreover, it discusses the implications of a brane-based ontology for spacetime emergence, providing a novel understanding of the relations between spatiotemporal and non-spatiotemporal structures.

10.1 Introduction: F-Theory and Philosophy

This Chapter examines the status of time and spacetime in the context of F-theory, a theoretical framework that extends Type IIB string theory beyond its perturbative description. F-theory is an important research program in theoretical physics for various reasons: it unifies elements from physics, geometry, and group theory; it gives a geometric description of a large portion of the string theory landscape; it brings to light many non-perturbative effects in the string coupling constant g_s; it offers valuable insights into string phenomenology, affecting our understanding of particle physics and quantum field theory, including aspects of Seiberg-Witten theory (Seiberg and Witten 1994a, b). Quoting Weigand (2018, 1):

> It is probably fair to say that its–in many ways ideal–location at the intersection of various dual M-theory corners makes F-theory the most generic currently controllable framework for studying string vacua in their geometric regime.

With more than two decades of progress, F-theory represents a mature line of research within the broader spectrum of string theory research. Thus, such a long-standing and developed field of study deserves philosophical exploration. A natural question that arises when one begins to explore philosophical issues in F-theory

is the question of the ontological and metaphysical implications of this theory. This issue becomes especially pressing when considering a unique feature F-theory presents—its apparent implication of there being *two* temporal dimensions. The paper aims to analyze this feature of F-theory and proposes an interpretation that vindicates a single temporal dimension underpinned by a brane-based ontology. Moreover, our discussion aims to highlight some of the outstanding philosophical issues that F-theory raises, particularly within its brane-based interpretation, and to point out some possible avenues for further philosophical research into this topic.

The Chapter is structured as follows: Sect. 10.2 introduces the critical aspects of F-theory that are relevant to our analysis of the two-time problem. Section 10.3 is dedicated to a detailed discussion about the nature of time in F-theory. Section 10.4 defends the project of developing a metaphysics of F-theory and broadly outlines the feature of the brane-based ontology, discussing some of the philosophical questions it raises and focusing on its implications for spacetime emergence. Finally, our conclusions are drawn in Sect. 10.5.

10.2 F-Theory in a Nutshell

F-theory, fundamentally, is a non-perturbative completion of Type IIB string theory,[1] which means that it is a theory that naturally reduces to[2] Type IIB string theory in a proper asymptotic limit. Drawing a parallel to illuminate the connection between F-theory and Type IIB string theory, one might consider the relationship between Newtonian mechanics and Special Relativity. Just as Special Relativity reduces to Newtonian mechanics in a specific asymptotic limit, notably the low-velocity limit ($v/c \to 0$), F-theory similarly reduces to Type IIB string theory within a certain

asymptotic limit. The goal of this section is to highlight some features of F-theory that are relevant to the main focus of this paper, i.e., understanding temporal dimension(s) in F-theory and discussing relations between spatiotemporal and non-spatiotemporal structures. One of the obstacles is that F-theory lacks a proper non-perturbative definition. However, F-theory can be approached and examined from multiple perspectives, each elucidating different aspects and features of the theory. We will provide a conceptually oriented introduction of F-theory from two different perspectives: first (Sect. 10.2.1) we provide an introduction to F-theory via duality with M-theory (Witten, 1996); then, in Sect. 10.2.2, we will outline the essential feature of an approach developed by (Cecotti, 2010), where F-theory is studied in terms of elliptic functions, an interpretation termed as *the elliptic reformulation of F-theory*.

10.2.1 F-Theory via Duality with M-Theory

M-theory (Witten, 1995) is an 11-dimensional theory that unifies the five different 10-dimensional string theories and is considered the fundamental candidate in the quest for a unified description of string theory. In particular, M-theory is supposed to be a unified non-perturbative description of the five perturbative superstring theories: Type IIA, Type IIB, Type I, Heterotic $E_8 \times E_8$, Heterotic $SO(32)$. In other words, M-theory provides the fundamental degrees of freedom which, in various limits, give rise to the 5 superstring theories. Moreover, the low-energy limit of M-theory is 11-dimensional supergravity, leading to the claim that M-theory should be, in some sense, an 11-dimensional theory rather than a 10-dimensional one like the perturbative superstring theories. The web of dualities that are supposed to give rise to M-theory is described in Fig. 10.1.

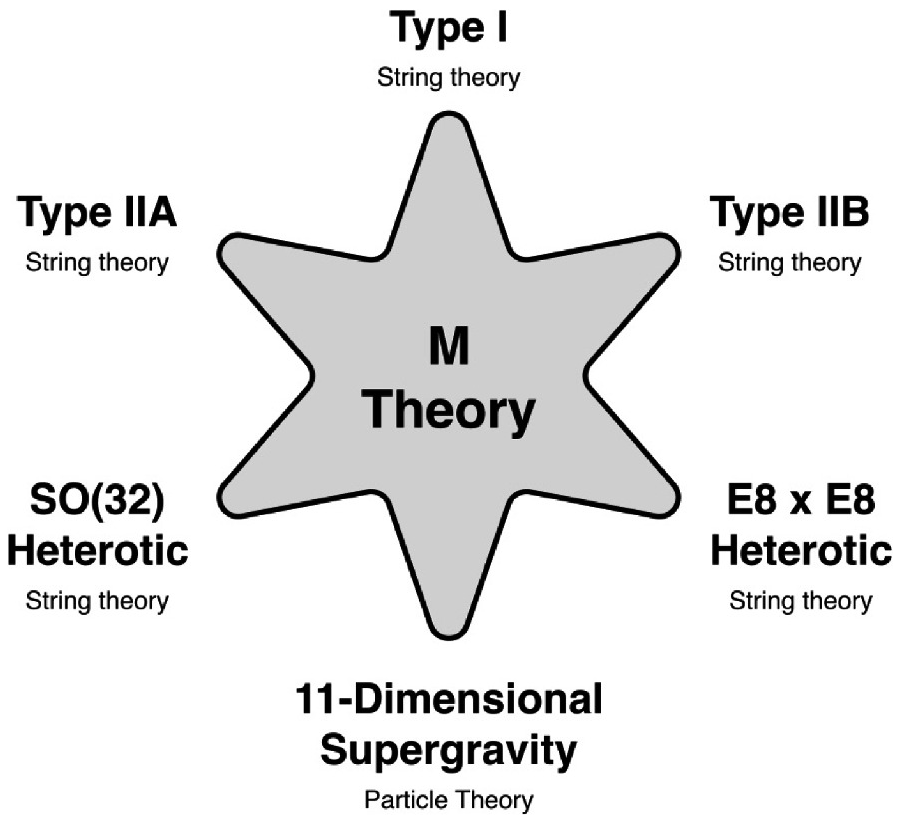

Fig. 10.1 This diagram represents the relationships between the different superstring theories and 11-dimensional supergravity. At the center, M-Theory is highlighted as providing a natural unification of the various string theories (Type I, Type IIA, Type IIB, $SO(32)$ Heterotic, and $E_8 \times E_8$ Heterotic) and of 11-dimensional supergravity. The 5 perturbative string theories, together with 11-dimensional supergravity, are expected to be particular limits (corners) of the more fundamental M-Theory.

A possible way to introduce F-theory, which was developed in Witten (1996), starts from M-theory and performs specific *duality* transformations and *compactifications*. Dualities, broadly speaking, are transformations that relate two different string theories. In particular, dualities establish whether or not two different string theories can be considered physically equivalent in a strong sense, including not only physical observables but also all theoretical quantities. By compactification, instead, we mean the process in which certain dimensions are compactified in such a way that they are too small to be observed at low energies.[3] Note that when we say *small*, we mean that the energy required to probe physical

phenomena involving these extra dimensions happens to be very high. This observation also clarifies the sense in which there can be scenarios where large extra dimensions are involved, as in *brane cosmology* (Brax & van de Bruck, 2003). Indeed, in these cases, what happens is that there are phenomena that are manifest at low energy scales and that are explained by, or related to, extra dimensions. Hence, given the connection between energy and length, it makes sense to say in this scenario that the extra dimensions are large.

Let us now see, step by step, how to obtain F-theory by performing compactifications and dualizations of M-theory.

- **Compactifying M-theory on a circle**. Take M-theory on a torus $S_1^A \times S_1^B$ and compactify one of the two circles, say S_1^A. In this way, one obtains type IIA supergravity.
- **T-dualize along the other circle**. T-duality is a particular kind of duality in string theory that relates, broadly speaking, a theory in which strings propagate on a circle of some radius R to a theory in which strings propagate on a circle of radius $1/R$. Apply T-duality to a compactified type IIA string theory along another circle S_1^B and obtain type IIB string theory. Therefore, type IIB string theory in $d = 10$ is obtained from 11d M-theory compactified on the torus $S_1^A \times S_1^B$.
- **Uplift**. F-theory comes into play by considering a 12-dimensional perspective where type IIB string theory just obtained is interpreted as the compactification of F-theory on a two-dimensional torus.

The procedure just outlined can be schematically represented by the following scheme:

$$\begin{array}{ccc} \textbf{M-theory in } d = 11 & & \textbf{F-theory in } d = 12 \\ \downarrow \text{compactification} & & \uparrow \text{uplift} \\ \textbf{IIA string theory in } d = 9 & \xleftrightarrow{T\text{-}duality} & \textbf{IIB string theory in } d = 10 \end{array}$$

In the next section, we will examine in more detail how F-theory and type IIB string theory are related. The goal of this discussion was to present a reasonably accessible picture of the relation between F-theory and M-theory. At the same time, however, the simplifications inevitably involved in such a task can lead to possible worries and misunderstandings regarding the exact nature of this relation. Let us clarify some of these possible misconceptions. The diagram above illustrates only a possible way to obtain F-theory from M-theory. However, on the basis of the diagram, one might worry that the relation between F-theory and M-theory is not a duality relation, and do so for two distinct reasons.

First of all, we constructed the claim regarding the mapping between F-theory and M-theory on the duality between Type IIA and Type IIB string theory, which does not necessarily imply a duality between M-theory and F-theory because Type IIA and Type IIB are just two limits of the non-perturbative M-theory and F-theory description, and we do not have, in principle, any decisive reason to believe that the duality is “uplifted” to the non-perturbative regime. Moreover, the reader may worry that the relation between F-theory and M-theory is not a duality relation because the diagram is one-directional, i.e., from M-theory to F-theory. But if dualities are symmetric, as is usually assumed, there should not be such a privileged direction.

These worries, however, can be shown to result from the specific simplifications of our exposition rather than from a fundamental flaw in the duality between M-theory and F-theory. Removing such simplifications, however, comes at the cost of significant complications in the story of the mapping between M-theory and F-theory. Indeed, our simplified description above does not tell the full story about the relation between M-theory and F-theory. The genuine duality between M-theory and F-theory is defined in a more technically precise, and likewise demanding,

fashion, as follows Cecotti (2010). An M-theory vacuum on the space $R^{1,2k} \times CY^{5-k}$,[4] where $CY^{5-k} \to K^{4-k}$ is an elliptic Calabi-Yau manifold (with a section), is dual to an F-theory vacuum with a 10-dimensional gravitational manifold $R^{1,2k+1} \times K^{4-k}$. More generally, if $\mathcal{N}$ is a Brinkmann space[5] with a Calabi-Yau flag,[6] elliptically fibered (with section) over the Brinkmann space N with a Kahler flag, then M-theory on $\mathcal{N}$ is dual to F-theory on $N \times \mathbb{R}$.

In a nutshell, then, given precise conditions on the geometry and more general mathematical properties of the spaces on which M-theory and F-theory are defined, there is a full duality relation between the two theories. In this sense, the two worries discussed above do not arise in the sophisticated description presented here since the duality relation is directly between M-theory and F-theory (and should not be "uplifted" from the duality between Type IIA and Type IIB string theory) and it is symmetric, i.e., the duality does not have a particular orientation.

10.2.2 The Elliptic Reformulation of F-Theory

Without delving too deeply into the intricacies of this reformulation, which would divert us from the central focus of this paper, we outline the features that are pertinent to our discussion. The central element of this reformulation is the *elliptic fibration*. Roughly, an elliptic fibration is a geometric structure that can be thought of as a space that locally looks like a torus over each point of another space (called the base space). In the case of F-theory, the 12-dimensional space can be seen as an elliptic fibration over a 10-dimensional base space.

This base space is the physical space in which type IIB string theory exists (as outlined in the previous section). Note that, from the perspective of type IIB string theory, it might be controversial to call the 10-dimensional spacetime (also called target space due to its role in the σ-model

description of perturbative string theory) the physical space of the theory. In particular, as argued for example in Huggett 2017, the existence of dualities that relate solutions of type IIB with different spacetime geometries casts doubt on the status of the 10-dimensional spacetime as the physical space. For the purposes of this article, we take the standard perspective that F-theory *geometrizes* these transformations and hence the Y_{12} manifold provides a duality invariant geometric description of the physical space of a solution of F-theory. Hence, it makes sense to speak of the 10-dimensional base space of the elliptic fibration as physical. Note that this statement only applies to those dualities that are actually geometrized in F-theory, i.e., S-duality. Other duality transformations, such as T-duality, might modify the verdict regarding which space is physical. However, for ease of exposition, we will avoid this complication and stick to pure F-theory without considering any further duality transformation.

In the elliptic reformulation of F-theory, the scalars' configuration[7] of type IIB string theory are understood as a (smooth) map which associates to each point x in spacetime $\mathcal{X}$ (the base space) an elliptic curve E_x, well-defined up to isomorphism. In particular, Cecotti (2010) shows that, in F-theory, the massless scalars' configuration is encoded in a (smooth) line bundle $L \to \mathcal{X}$, and two sections $A(x)$ and $B(x)$ trough the elliptic curve E_x represented by the following Weierstrass curve:

$$Y^2Z = X^3 + A(x)XZ + B(x)Z^3 \ . \tag{10.1}$$

We are interested in this very specific detail of the elliptic reformulation of F-theory because it underpins an important conceptual point: (10.1) describes a twelve-dimensional manifold $\mathcal{Y}_{12}$ with a natural projection to the ten-dimensional spacetime $\mathcal{X}$,

$$\pi : \mathcal{Y}_{12} \to \mathcal{X} \ .$$

The map π is formally termed a *fibration*, represented by Eq. (10.1). The fibration π possesses a preferred *section* $\sigma : \mathcal{X} \to \mathcal{Y}_{12}$,[8] established by mapping the point $x \in \mathcal{X}$ to the chosen point $\mathcal{O}_x$ on the elliptic curve E_x over x. This preferred section σ allows for a mapping between the spacetime $\mathcal{X}$ and the manifold $\mathcal{Y}_{12}$. One can thus identify the spacetime $\mathcal{X}$ with its image under the preferred section σ: the spacetime is seen as a submanifold of $\mathcal{Y}_{12}$. Note that one should not confuse this description with the typical σ-model[9] description of perturbative string theory.[10]

In σ-models, fields are maps from the worldsheet, which is a 2-dimensional manifold that represents the string as it evolves, and a base manifold, also called the target, which is a higher-dimensional (in perturbative string theory 10-dimensional) spacetime manifold in which the string propagates. These fields, i.e., maps from the worldsheet to the target space, upon quantization, give rise to a 2-dimensional quantum field theory with conformal symmetry, i.e., a conformal field theory. This theory is usually called the *worldsheet conformal field theory*. The F-theory description presented here does not have the structure of a σ-model, as it is a non-perturbative description. Consequently, the concept of target space is not applicable in this context, and neither $\mathcal{X}$ nor $\mathcal{Y}_{12}$ should be considered as target spaces within the framework of F-theory. However, as we did above, $\mathcal{X}$ can be considered a target space from the point of view of type IIB string theory, i.e., in the perturbative regime of F-theory. In this framework, F-theory can be understood as a theory over an ordinary ten-dimensional manifold $\mathcal{X}$, representing the physical spacetime, or as a theory over the twelve-dimensional spacetime $\mathcal{Y}_{12}$. Gravity, described by the geometry of the spacetime manifold $\mathcal{X}$, and the scalars' dynamics are unified in the geometry of the twelve-dimensional manifold $\mathcal{Y}_{12}$.

The physical import of this claim and the previous ones might not be immediately clear. A heuristic approach to discern the physical picture underlying these concepts and to understand how F-theory is a non-perturbative completion of Type IIB is the following: type IIB is self-dual, i.e., solutions to Type IIB string theory are related by S-duality, a duality which relates a weakly coupled, perturbative description, to a strongly coupled, non-perturbative one, and vice versa; in its most simple examples, S-duality maps a theory with coupling constant g to one with coupling constant $\frac{1}{g}$, effectively inverting the coupling constants of the two theories. In more complicated examples, these transformations of the coupling constants can be represented in terms of the action of the group $SL(2,\mathbb{Z})$. For instance, the coupling constant g of the theory is promoted to a complex massless scalar τ transforming as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{10.2}$$

where the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an $SL(2,\mathbb{Z})$ matrix, i.e., it is a 2×2 matrix, whose component are integer numbers and with determinant one.[11] Moreover, S-dual solutions can be understood to be physically equivalent. However, due to its perturbative nature, this equivalence is not *explicit* in Type IIB: solutions related by S-duality are not manifestly equivalent within the framework of Type IIB string theory. Note that by manifestly equivalent, we mean a situation in which the physical content of the theory is expressed in terms of physical quantities invariant under the relevant (symmetry and/or duality) transformations.

F-theory, a non-perturbative formulation of Type IIB string theory, studies S-duality in its most general form as an invariance under the $SL(2,\mathbb{Z})$ group, as we mentioned above. The elliptic fibration of F-theory precisely embodies

this invariance under the $SL(2,\mathbb{Z})$ group, meaning that two points on the elliptic fibration are connected with an $SL(2,\mathbb{Z})$ transformation. Heuristically speaking, a fibration can be understood as a gauge field (Ortín, 2004). Two points on an elliptic curve are connected through an $SL(2,\mathbb{Z})$ transformation in the same way two points on a gauge fiber are connected through a gauge transformation. An elliptic fibration is, thus, a geometrization of the duality invariance under $SL(2,\mathbb{Z})$ of Type IIB string theory. In the traditional lore of gauge fields, this means that configurations connected by such transformations are manifestly equivalent.

In the last part of this section, let us discuss why F-theory, as a theory over the twelve-dimensional spacetime $\mathcal{Y}_{12}$, raises interpretational problems. F-theory is a (locally) supersymmetric theory with 32 real supercharges, i.e., a theory of supergravity. By supergravity, we mean a supersymmetric extension of General Relativity. Any theory in which supersymmetry is a *local*, i.e., gauged, symmetry is a supergravity theory.[12] From the classification of spinors in spaces of signature (p,q), one can argue that within a twelve-dimensional Minkowski space, characterized by a signature $(11,1)$, the minimal spinor has 64 real components. Consequently, no covariant theory can have 32 supercharges in twelve dimensions, as F-theory requires.

Thus, $\mathcal{Y}_{12}$ cannot be understood as a conventional (meaning with signature $(11,1)$ as in Minkowski spacetime) twelve-dimensional spacetime. On the contrary, Majorana-Weyl spinors exist in a space with signature $(10,2)$ and have 32 real components. Thus, $\mathcal{Y}_{12}$ can be understood as a manifold with signature $(10,2)$, suggesting that the fundamental twelve-dimensional theory may live within a spacetime with signature $(10,2)$, thus involving the presence of *two* temporal dimensions. However, such a

theory would be troublesome because it casts shadows of uncertainty on foundational physical doctrines such as causality, unitarity, and the second law of thermodynamics. In particular, as discussed in Tegmark (1997), the existence of two times does not allow the formulation of a well-defined initial value problem, hence making the theory not predictive.[13]

10.3 Time in F-Theory

We have seen in the previous section the basic structure of F-theory, understood as a non-perturbative completion of type IIB string theory. In particular, we have seen the heuristic picture whereby we can think of F-theory as geometrizing the duality transformations that are not manifest in type IIB string theory. These transformations are represented in terms of an elliptic fibration, i.e., in terms of a fiber bundle where each fiber is an elliptic curve. Hence, F-theory can be seen, heuristically, as a combination of type IIB string theory and an elliptic fibration, as is evident from the discussion in Sect. 10.2.

As we also mentioned in the previous section, F-theory seems *prima facie* to entail the existence of two different time directions. This two-times idea is both technically and conceptually extremely challenging insofar as it leads to deep questions regarding the consistency of a theory with two times and the physical and metaphysical interpretation of such claims concerning the existence of multiple time dimensions.[14]

Nonetheless, on closer inspection, it is unclear whether or not F-theory requires the existence of two-time dimensions. At this juncture, it is helpful to be more precise regarding what we mean in this context when we say we have a time dimension. By a time dimension, we mean a spacetime direction with a minus sign in the spacetime

signature.[15] Hence, as discussed, an (11, 1) spacetime would be a spacetime with 11 space dimensions and one time dimension, while a (10, 2) spacetime would be one with ten space dimensions and two time dimensions. Hence, the claim that there is a time dimension, at least in the broad relativistic framework that underlies this discussion, involves the conjunction of two statements: (i) there must be a spacetime dimension, and (ii) this dimension must have a minus sign in the spacetime signature.[16] Note that the signature of spacetime is a global property of the manifold and, hence, does not depend on a particular slicing of the manifold. What does depend on the slicing of the manifold is, given a choice of coordinates, which of these coordinates is going to have a minus sign. However, our definition does not depend in any way on this latter observation. A particular dimension will count as a time dimension only if it satisfies both conditions, while failure to satisfy either one will mean that we are not dealing with a time dimension but with something else. In particular, failure of (i) entails that the relevant degrees of freedom propagating along said dimension do not count as spatiotemporal degrees of freedom since the dimension in question is not spatiotemporal.

The easiest way to see how a dimension might fail to be temporal is to think of spatial dimensions. Spatial dimensions satisfy (i) because they are spacetime dimensions but fail to satisfy (ii) because they do not appear in the spacetime signature with a minus but a plus sign. However, another way a certain dimension might fail to be time would be to fail to satisfy (i) rather than (ii). Hence, we would be dealing with a dimension having a minus sign in the signature but failing to count as a spacetime dimension, and hence being inadequate to count as a time, at least under the plausible assumption that time

is spatiotemporal and that something not spatiotemporal cannot be temporal. What criterion should we then use to decide whether or not a dimension counts as spatiotemporal? This question can, in principle, be extremely fraught, and it is certainly the case that a great deal of philosophical work has been done to identify under what conditions something can count as spacetime. Arguably, the most advanced attempts in this direction have been made within the spacetime functionalism program (Knox, 2013; Knox, 2019; Baker, 2021), where the question of under which conditions a certain structure counts as spatiotemporal becomes especially important since the answer corresponds to identifying the functional role of spacetime, and hence to a crucial step in fulfilling the spacetime functionalism program. Recall that, as discussed in Sect. 10.2.2, $\mathcal{Y}_{12}$ provides a duality-invariant geometric description of the physical space of a solution of F-theory. Hence, the question that functionalism is tasked to resolve is whether or not all twelve dimensions of $\mathcal{Y}_{12}$ count as spatiotemporal. Note that, as we mentioned above, we will not take into account any further duality transformation that might apply to $\mathcal{Y}_{12}$ besides the S-duality that F-theory geometrizes since we are only concerned with F-theory and not any further extension of F-theory.

In this paper, we will not delve into this rich and complicated debate for two reasons: first of all, because doing so would require a separate contribution in and of itself, something which goes beyond the scope of the present work; second of all, because the broadly geometric framework that is employed in these debates is difficult to generalize to a theory like F-Theory, which is as much quantum field theory and particle physics as it is general relativity and spacetime; finally, because what we are ultimately interested in is the question whether a certain dimension counts as spatiotemporal, not of whether any structure in general does. Hence, the problem we are

interested in is much more restricted and does not require the sophisticated criteria proposed in the spacetime functionalism literature. For these reasons, we will instead rely on a particle physics-inspired criterion, which will allow us to more easily identify spatiotemporal structure in F-Theory while at the same time retaining basic intuitions regarding what should count as spatiotemporal and what should not.

The criterion we have in mind is the following:

(SD) Spacetime Dimension: a specific dimension in a manifold $\mathcal{M}$ counts as spacetime iff the graviton propagates along that dimension.

Some remarks are in order. The basic gist of this definition is that if the graviton, i.e., the particle which encodes excitations of the gravitational field, propagates along a particular dimension, then that dimension is spatiotemporal. This choice is due to the general relativistic identification of spacetime with gravitation, which means that the graviton encodes excitations of the spacetime metric itself. More precisely, the relation between the graviton and the spacetime metric can be made explicit by looking at how gravitons emerge in the context of perturbation theory for general relativity treated as quantum effective field theory.[17]

In this context, gravitons emerge as (quantum field theoretic) perturbations, hence excitations, of the general relativistic spacetime background around which perturbation theory is constructed; in this sense, gravitons emerge as excitations, or *quanta*, of the gravitational field, in much the same way in which any other quanta emerge from their respective quantum fields: for example, the photon emerges as a quantum of the quantum electromagnetic field in quantum electrodynamics in exactly this way. Hence, just like the photon encodes excitations of the quantum electromagnetic field, the

graviton encodes excitations of the gravitational field, hence of spacetime. To further strengthen this association, we can think about the scattering of gravitons, i.e., the basic dynamical quantity of general relativity treated as an effective quantum field theory. Such scattering amplitudes between gravitons are computed via the gravitational path integral, i.e., without delving into excessive formal details, a (purely formal) integral[18] over the space of metrics (and possibly matter fields) of general relativity, weighted by (the exponential of) the Einstein-Hilbert action, which itself is a function of the spacetime metric (plus possibly matter fields). Hence, we see again that the properties of gravitons are intimately related to the properties of spacetime in general relativity, as they are encoded in the Einstein-Hilbert action and more generally in the space of metrics of general relativity.

The upshot of this observation is then that if the graviton does not propagate along a particular dimension, there is no spacetime metric, and hence no spacetime, along that direction. We say this because spacetime itself does not extend along that direction; otherwise, it would be possible to excite the metric field in that direction and, hence, for the graviton to propagate. An interesting feature of this criterion is that, since gravitons make perfect sense also at the quantum level, this criterion is applicable to theories of quantum gravity and in particular, is applicable to F-Theory.

With this precise criterion for the spatiotemporality of a certain dimension, let us go back to our original question of whether or not F-Theory really has two times. Here, an especially illuminating discussion of this issue is given by Cecotti (2010) . There, we start from the observation that the fibration π splits the manifold $\mathcal{Y}_{12}$ into $\mathcal{X}$, as remarked above, with signature (9, 1), and an elliptic fiber, which, as a manifold, has a signature (1, 1). While, as remarked

above, this leads us to an apparently two times theory over a manifold $\mathcal{Y}_{12}$ with signature (10, 2), Cecotti, (2010) points out that a more careful inspection of the physics casts doubts on these claims. In particular, it is the case that the graviton, in F-Theory, only propagates along $\mathcal{X}$, and not along the two dimensions spanned by the elliptic curve. As such, the (1, 1) signature manifold representing the elliptic fibration does not count as spacetime. One can see this especially clearly from the point of view of **(SD)**, the criterion for spatiotemporality that we discussed above. Since the graviton does not propagate along the elliptic fibration, the spacetime metric does not make sense along those directions. Hence, if we take seriously general relativity's insight that the geometry of the spacetime metric gives spacetime, there is no spacetime along the elliptic fibration.[19]

This observation is crucial for discussing whether or not F-Theory has two times. For recall, our two conditions (i) and (ii) for a certain dimension count as a time dimension. The fact that the graviton does not propagate along the elliptic fibration implies that one of the minus sign dimensions in $\mathcal{Y}_{12}$ is not a spacetime dimension. Hence, it fails to satisfy (i). As such, there is only one time dimension in F-Theory, the one with a minus sign in $\mathcal{X}$, because that second candidate for a time dimension, on closer inspection, fails to be spatiotemporal. The minus-sign dimension in the elliptic fibration, then, rather than being a time dimension, encodes some non-spatiotemporal, and less than spatiotemporal, features of the geometry of $\mathcal{Y}_{12}$, in particular features related, in physical terms, to the invariance of F-theory under S-duality transformations. Hence, the elliptic fibration should be seen as a genuine component of the non-spatiotemporal structure making up the fundamental degrees of freedom of F-theory, one that, moreover, does not have any spatiotemporal limit or

spatiotemporal counterpart in some type of low-energy/classical limit; because in this limit the theory becomes type IIB string theory, for which, as we discussed above, S-duality does not act manifestly, i.e., type IIB string theory is not S-duality invariant. This means that solutions of type IIB string theory lack the structure of the elliptic fibration, and S-duality is implemented as a map between different solutions, not one with respect to which a single solution is invariant.

In conclusion, then, we have seen how in F-Theory, the appearance of two-time dimensions, on a closer look, leaves room for a much richer and more sophisticated geometrical and physical picture, which involves a seamless intertwining of spatiotemporal and non-spatiotemporal aspects. In particular, this picture seems to naturally involve the idea that spacetime is best understood as a dynamical entity constrained to propagate on the 10-dimensional submanifold of $\mathcal{Y}_{12}$ defined by $\mathcal{X}$, while $\mathcal{Y}_{12}$ is best viewed as some kind of non-spatiotemporal entity. In string theory, more generally, and F-theory, more specifically, this kind of structure defined by a submanifold carrying dynamical degrees of freedom constrained to propagate on it is called a *brane* and plays a crucial role in understanding the non-perturbative dynamics of string theory.[20] This observation then, which is a direct consequence of the argument that F-theory does not have multiple time dimensions discussed in this section, leads to the following natural suggestion (Vafa, 1996; Cecotti, 2010):

> The ontology of F-theory, rather than concerning strings propagating in some target spacetime, is given by branes, i.e., higher-dimensional extended objects, propagating along a non-spatiotemporal entity described by the geometry of $\mathcal{Y}_{12}$.

The next section will discuss some aspects of this fascinating picture.

10.4 Metaphysics of F-Theory

We have seen in the previous section how F-Theory, rather than a theory of 12-dimensional spacetime with two-time dimensions, is better understood as a 12-dimensional theory that includes a 10-dimensional spacetime with only a one-time dimension. In reaching this conclusion, a crucial role was played by Cecotti (2010)'s result that the graviton does not propagate along the elliptic fibration. However, not enough has been said thus far regarding how to interpret this state of affairs, and in particular on the metaphysical picture underpinning such claims. In this section, we will first of all argue in defense of the need to develop such a metaphysical picture of F-theory, despite the theory's novelty and speculative nature (Sect. 10.4.1). Following this, we will discuss a suggestion made in Cecotti (2010) to interpret F-theory in terms of branes, and examine its implications for discussions of spacetime emergence (Sect. 10.4.2).

10.4.1 In Defense of a Metaphysics of F-Theory

To start, let us note that focusing on the nature of the twelve-dimensional space $\mathcal{Y}_{12}$ in F-theory gives rise to many intriguing questions: for example, is $\mathcal{Y}_{12}$ a genuine physical entity, or merely a mathematical artifice deployed for computational convenience? Given the absence of a complete formulation of F-theory, any attempt towards resolving this question must remain tentative. It is, however, interesting to ask how general approaches to the ontology of duality, such as those surveyed in the taxonomy of Le Bihan and Read (2018), can bear on such a question, and vice versa what F-theory has to say about such approaches; while nonetheless remembering the essentially

tentative character of these considerations, given the still not fully fleshed out nature of F-theory we alluded to above.

Indeed, F-theory, by giving a non-perturbative formulation of type IIB string theory, sits at the center of the web of dualities between the various superstring theories detailed in Fig. 10.1; for this reason, it is reasonable to expect F-theory to have much to say regarding the ontology underlying such dual models, and in particular regarding the degrees of freedom which give rise to the various superstring theories. Indeed, from this point of view, the study of F-theory would be a natural avenue to understand which of the ontological stances towards dualities that Read, (2018) identify should be preferred in non-perturbative string theory; in particular, it seems natural to think of F-theory as favoring either the overarching theory approach or the common core approach, as identified in f Le Bihan and Read (2018).[21]

At the same time, we know that F-theory cannot be the whole story regarding the full, non-perturbative formulation of string theory: F-theory does not really capture the same degrees of freedom as M-theory since we do not have a duality invariant formulation of the two theories. Indeed, as we discussed before, F-theory is duality-invariant only with respect to S-duality, but string theory involves many more dualities between the superstring theories, starting, for example, with T-duality. F-theory does not capture any duality-invariant content with respect to T-duality. Hence, F-theory cannot be the whole story regarding the non-perturbative, duality-invariant content of string theory, and more is needed to fully understand the nature of M-theory/non-perturbative string theory. In particular, a natural question to ask in this regard would be how to understand the ontology of F-theory *vis à vis* the other string dualities, with respect to

which F-theory is not invariant; this question, however, as we indicated above, is one that, we will not pursue.

Nevertheless, these considerations might prompt the reader to a skeptical reaction regarding the project we are undertaking in this paper: what is the point of studying the ontology of F-theory, as we are doing, if we already know that it is not going to be the full story regarding non-perturbative string theory, and in particular that whatever ontology we derive from the study of F-theory is going to be modified by taking into account the other string dualities not included in F-theory, and the results coming from the study of M-theory? The answer to this question is twofold:

- First of all, the fact that F-theory does not provide the most fundamental description possible of the ontology of non-perturbative string theory does not imply that clarifying the ontology of F-theory as a non-fundamental theory is a worthless task[22]; indeed, as things currently stand, all discussions of the ontology of physical theories concern similarly non-fundamental theories, starting with quantum field theory and general relativity, who cannot claim any fundamental status since they both concern a limited domain of application.[23] Moreover, all such theories currently studied are less fundamental than F-theory, which, together with the corners of M-theory currently known, is the most fundamental theory we currently possess.[24] For these reasons, it seems perfectly natural to want to study the ontology of F-theory since, at least, F-theory is going to give us the most general and fundamental ontological perspective currently available within string theory while we wait for the development of a full non-perturbative formulation of string theory.
- Second of all, the study of the ontology of F-theory can play an important heuristic role in the development of a non-perturbative theory of strings and its ontology; in this respect, we have in mind that F-theory could play with

respect to the full non-perturbative formulation of string theory a role similar to the *heuristic role* for dualities described by De Haro (2019). In particular, much like dualities can provide clues and tools to construct a more fundamental theory underlying them, F-theory is a natural place to try to understand and get clues for the full non-perturbative formulation of string theory; indeed, F-theory, as we currently understand it, should capture at least a sector of such non-perturbative theory, and so has a natural role as a heuristic tool in theory construction. In this context, then, understanding the ontology of F-theory becomes an even more interesting task since it allows us to isolate the relevant conceptual and ontological innovations that are characteristic of f-theory, thus giving us fresh ideas and perspective towards the basic conceptual and ontological structure that should underlay a full non-perturbative formulation of string theory. In articulating the ontology of F-theory, or at least in articulating the basic ontological options that F-theory seems to suggest, we take ourselves then to be contributing to such a project, whose ultimate goal is to give a good conceptual scheme in which non-perturbative string theory can be developed, free of the misunderstandings that an excessively naive ontological picture might lead to. And in this task, reasoning on the theories that we already have and that we know to capture some aspects of this yet unknown theory, as F-theory does, appears to us an invaluable tool, and to make worth the exploration of the ontology of F-theory.

For these reasons, we take that understanding the ontology of F-theory is a valuable task, despite the necessarily provisional nature of this endeavor, both due to the limitations in our current understanding of F-theory and to its non-fundamental nature.

10.4.2 Steps Towards the Metaphysics of F-Theory

Let us now go back to the question of the ontology underlying F-theory. To better understand the theory's ontology, it is useful to remember an observation that we made in Sect. 10.2. There, we noticed that $\mathcal{X}$, the spacetime manifold encoding the data of type IIB string theory in F-Theory, can be understood as a submanifold of $\mathcal{Y}_{12}$. This type of construction, i.e., a submanifold carrying dynamical information that is confined to propagate only in that submanifold, has an immediate interpretation in string theory: the submanifold $\mathcal{X}$, carrying the data of type IIB string theory, is a 10-dimensional *brane*, i.e., a higher dimensional extended object typical of string theory.[25] Branes, in general, can carry all sorts of dynamical and physical information, from gauge fields to a dynamical spacetime metric, as in the case of $\mathcal{X}$. In string theory, the claim that a brane carries a certain field means that that field will propagate only in the dimensions spanned by that brane and will not be present in the others transverse to the brane. In other words, a field carried by a brane is confined to that brane. So, in particular, the spacetime metric field is confined to the 10-dimensional brane defined by the submanifold $\mathcal{X}$.

More generally, as remarked by Cecotti (2010), branes of different dimensionalities carry different degrees of freedom in F-Theory, where the dimensionality of the brane is fixed by the different spins characteristic of those degrees of freedom, according to the following classification[26]:

- Matter fields, which have the lowest spin, i.e., spin $\frac{1}{2}$,[27] live on 6-dimensional submanifolds, i.e., 5-branes.
- Gauge fields, which have a higher spin than matter fields, i.e., spin 1, live on 8-dimensional submanifolds, i.e., 7-

branes.

- The gravitational field, and hence spacetime, which has an even larger spin, i.e., spin 2, lives on the 10-dimensional submanifold $\mathcal{X}$, i.e., a 9-brane.

This classification leads to a picture of the ontology of F-theory where F-theory, rather than a theory of strings or particles carrying the fundamental degrees of freedom of the theory, is a theory whose degrees of freedom are all characterized and carried by branes of different dimensionalities, propagating in a 12-dimensional non-spatiotemporal[28] geometry. Notice that when we say that $\mathcal{Y}_{12}$ is non-spatiotemporal, what we mean is that there is no such thing as a global description of $\mathcal{Y}_{12}$'s geometry which is spatiotemporal, which can be seen by the fact that there is no metric field that can cover all the 12 dimensions of $\mathcal{Y}_{12}$. This fact is a direct consequence of the argument in Sect. 10.3 that the two directions of the elliptic fibration are non-spatiotemporal.

Nonetheless, it is the case, as we have emphasized at various junctures in this paper, that there is a subspace of $\mathcal{Y}_{12}$ that is spatiotemporal, i.e., the subspace defined by $\mathcal{X}$; this fact can be seen directly from the observation that there is a metric field which covers the 10 dimensions described by $\mathcal{X}$. Indeed, this metric field is none other than the metric field of type IIB string theory. In this sense, then, when we say that $\mathcal{Y}_{12}$ is non-spatiotemporal, what we mean is that $\mathcal{Y}_{12}$ is in general non-spatiotemporal, i.e., that it does not have a global spacetime description, not the stronger claim that there are no spatiotemporal descriptions even for any subspace of $\mathcal{Y}_{12}$; indeed, such a stronger claim would e false since $\mathcal{X}$ exists. Rather, the point is that the geometry of $\mathcal{Y}_{12}$ does not admit a complete description in spatiotemporal terms but rather only a partial one, restricted to the subspace defined by $\mathcal{X}$; this fact is encoded in F-theory via the description of the

theory's ontology in terms of branes, which include the 9-brane defined by $\mathcal{X}$, propagating on the (in general) non-spatiotemporal geometry defined by $\mathcal{Y}_{12}$. The physics of F-theory is then a consequence of the interactions among these various branes and between the branes and $\mathcal{Y}_{12}$.

The picture that emerges is one where there are a variety of extended objects on which the degrees of freedom of F-Theory are constrained to move. The interactions between these extended objects, which are, at least in F-theory, the fundamental building blocks of the theory, give rise to the overall geometry that characterizes a configuration of F-theory and hence describes its physics. The theory, then, is a theory of 12-dimensional non-spatiotemporal geometry[29] on which a variety of possibly spatiotemporal degrees of freedom propagate. The interaction between these degrees of freedom, constrained to move on a specific brane, and the underlying 12-dimensional non-spatiotemporal arena in which F-theory lives is the origin for the rich and beautiful geometry that F-theory ultimately deals with and characterizes its solutions.

This kind of picture immediately raises various deep and fascinating philosophical questions. First, one might wonder what the exact status of the branes should be: we have described them as extended objects that encode the fundamental degrees of freedom of F-theory; should we think of them as extended simples or as composed of more fundamental parts? In the context of string theory, the possibility of thinking about strings and branes in terms of extended simples has been criticized by Baker (2016), though his discussion is ultimately more focused on string theory rather than F-theory. A more general argument regarding the possibility of extended simples in quantum gravity is offered by Baron and Le Bihan (2022b). In particular, as they argue, whether or not branes (or any

other entity found in a quantum gravity theory) count as extended simples will depend on whether or not they are objects extended in spacetime; hence, in the context of F-theory, if we count the $\mathcal{Y}_{12}$ manifold as non-spatiotemporal as we argued above, it is unclear how to count branes in F-theory as spatiotemporal, and hence as extended simples. In particular, either the branes will not be extended at all, or they will be extended only in a non-spatiotemporal sense, in both cases leading to a tension with the standard idea of extended simples.[30]

On a similar note, how can we properly conceive of locality in a theory including extended objects, given that extended objects cannot preserve the Lorentz symmetry, which usually defines locality for relativistic theories? And finally, how should we interpret the web of dualities of F-theory is at the center of, as a geometrization of S-duality, and which eventually is supposed to relate F-theory to the even more elusive M-theory? These are all deep and important philosophical questions for which a complete treatment would require a book rather than an article. For this reason, we will leave them for future work and instead focus, in the remainder of this paper, on the relation between the brane ontology that we are describing and spacetime emergence.

A particularly interesting feature in the context of spacetime emergence is the fact that the spatiotemporal and non-spatiotemporal aspects of the theory, instead of being confined to different energy regimes, as is the case in most examples discussed in the philosophy literature,[31] interact together to give rise to a configuration of F-theory. Indeed, this interaction between spatiotemporal and non-spatiotemporal is one of the fundamental features of F-theory. As discussed in Sect. 10.2, the geometrization of S-duality, which is at the core of F-theory, relies on the fact that a purely non-spatiotemporal component, i.e., the

elliptic fibration (see Sect. 10.3 for our argument for this claim), interacts and combines with the spatiotemporal physics describes by $\mathcal{X}$, which is, broadly speaking, Type IIB string theory.

This feature of F-theory points towards a very different sense in which we can speak of spacetime emergence, one where the fundamentality status of spatiotemporal structure lies on a par with that of non-spatiotemporal structure. In particular, both spatiotemporal and non-spatiotemporal structures contribute equally to determining the properties of a certain F-theory model, and there is no obvious sense in which, for example, spacetime is composed of non-spatiotemporal parts (Baron & Le Bihan, 2022a), or in which non-spatiotemporal degrees of freedom functionally realize spacetime (Lam & Wüthrich, 2018). Rather, in F-theory, the spatiotemporal structure is not necessarily less fundamental than the non-spatiotemporal structure but less general. Less general in the sense that the non-spatiotemporal concepts can be applied anywhere on the 12-dimensional manifold $\mathcal{Y}_{12}$, while spatiotemporal concepts only make sense when we restrict our attention to the 9-brane given by $\mathcal{X}$. As an example of this phenomenon, we can compare the status of metric and complex structure[32] in F-theory. While, as remarked above, the metric structure only makes sense along 10 of the 12 dimensions of F-theory, i.e., those described by $\mathcal{X}$, the complex structure, which in principle does not have a clean spatiotemporal interpretation, makes sense in all the 12 dimensions of $\mathcal{Y}_{12}$. Hence, the non-spatiotemporal structure enjoys greater generality in terms of scope compared to the spatiotemporal one rather than being constitutive of the spatiotemporal structure as usually assumed in the spacetime emergence literature.

One might try to make sense of this state of affairs in mereological terms, along the lines of the proposal of Baron

and Le Bihan (2022a). In particular, the suggestion would be that the 9-brane $\mathcal{X}$ is a proper part of the non-spatiotemporal structure of $\mathcal{Y}_{12}$, which *prima facie* leads to the possibility of understanding the relation between non-spatiotemporal and spatiotemporal degrees of freedom in terms of the former composing *non-spatiotemporally*, i.e., by taking fusions of non-spatiotemporal atoms, the latter. The origin of this suggestion lies in the aforementioned proper part relation between $\mathcal{X}$ and $\mathcal{Y}_{12}$, which appears to open the way to the use of mereological concepts in F-theory. The problem with this suggestion, however, is that while it makes sense to think of $\mathcal{X}$ and $\mathcal{Y}_{12}$ as standing in a proper part relation, it is highly unclear how to think of the degrees of freedom of $\mathcal{Y}_{12}$ as composing $\mathcal{X}$. We say this because $\mathcal{X}$ is ultimately a 9-brane that exists and propagates on the non-spatiotemporal $\mathcal{Y}_{12}$, rather than being composed by it; as things stand in our understanding of F-theory, then, there is no obvious mereological understanding of the relation between $\mathcal{X}$ and $\mathcal{Y}_{12}$, as we remarked above, and new ideas are needed to understand this relation. Whether or not these ideas will have a mereological component, as suggested by the proper part relation between $\mathcal{X}$ and $\mathcal{Y}_{12}$, is a topic for future research.

Therefore, how to exactly understand this specific form of the relation between spatiotemporal and non-spatiotemporal structure appears as an awesome, and awesomely rewarding, challenge for philosophers of physics; a challenge which should especially interesting given the deep ties between F-theory and phenomenological aspects of quantum gravity and string theory, and also given the relation between F-theory and issues having to do with the ontology and interpretation of dualities, which take a purely geometrical form in F-theory. However interesting these questions are, their solution goes beyond the scope of this paper, whose main goal was

to outline these problems rather than solve them. We will then leave the solution of these outstanding issues to future work.

10.5 Conclusions

In this paper, we analyzed the status of time and spacetime in the context of F-theory. First, we defined what we mean by *time dimension*, i.e., a dimension that both (i) exists in spacetime and has (ii) a negative signature. We argued that, in F-theory, we would need a criterion to understand if a given dimension exists in spacetime since F-theory is not a simple theory of gravitation but also includes quantum mechanical effects. We thus defined what we mean for a dimension to exist in spacetime **(SD)**, i.e., dimensions along which the graviton (which represents spacetime) can propagate.

We thus used the propagation of the graviton as a benchmark to determine what counts as spacetime in F-theory. F-theory can be *prima facie* understood as a theory with two-time dimensions since it has two dimensions with a negative signature. However, there is only one time dimension in F-theory since gravitons can propagate only along one of the two negative-signature dimensions. This reshapes our understanding of F-theory not as a model with two times but as a sophisticated model with one time. Moreover, we argued that F-theory's ontology can be described using a variety of extended objects (branes) on which the degrees of freedom of F-Theory are constrained to move. The interplay of these branes within a 12-dimensional manifold gives rise to the rich structure and dynamics that characterize F-theory. We also discussed some novel aspects of F-theory concerning spacetime emergence. In particular, we discussed how spatiotemporal and non-spatiotemporal structures are equally fundamental

but have different scopes. In particular, non-spatiotemporal objects are universal across F-theory's 12-dimensional manifold, while spatiotemporal entities are limited to a 10-dimensional brane.

We showed that this understanding of the interplay between spatiotemporal and non-spatiotemporal structures contrasts with traditional views on spacetime emergence. Understanding the details of the relation between spatiotemporal and non-spatiotemporal structures in F-theory represents a challenging and important challenge for philosophers of physics, with implications for understanding quantum gravity and string theory.

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Footnotes

1 Proofs and technical details regarding the claims in this section can be found in Cecotti (2010) and references therein. Cecotti (2010) provides a superb introduction to F-theory, while the original paper on F-theory is Vafa (1996). For technical background on string theory, see Green et al. (1987a, 1987b).

2 Here we use the physicist convention of calling the relation between a more fundamental and less fundamental theory *reduction*. In philosophical parlance, this would be more properly called *emergence*, and we will indeed speak of emergence later on in the paper when talking about *spacetime emergence*.

3 For a philosophical discussion on the significance of compactifications in string theory see Cinti and Sanchioni 2022.

4 CY^{5-k} is a Calabi-Yau manifold, which is a compact, complex manifold with zero Ricci curvature. $5-k$ is a dimensional specification, where *k* is a parameter that adjusts the dimensionality of the manifold in question.

5 A Brinkmann space is a Lorentzian space with some specific condition on its holonomy, i.e., that it should preserve a null vector.

6 A flag in a vector space is a nested system of linear subspaces, one of each dimension from 0 to $n-1$, where *n* is the dimension of the vector space.

7 When we speak of scalars here, we mean the parameters that identify specific solutions of the theory. In string theory, and hence in F-theory, these parameters, also called moduli, are not free parameters but are instead represented by dynamical scalar fields whose equations of motion are part of the theory itself. As such, we only speak of scalars.

8 A section is a continuous mapping that assigns to each point in the base space a point in the total space. A preferred section is a particular choice of

such a section, often used as a reference or starting point for further constructions or analyses.

9 Note that the σ in σ-model should not be confused with the σ of the preferred section in the previous paragraph. There is no relation between the two notions except for an unfortunate coincidence of notation.

10 For background on σ-models, see Hull (1987), Callan and Thorlacius (1988). Perturbative string theory, i.e., the use of σ-models to describe quantum gravity via the perturbative dynamics of strings, is introduced, as mentioned above, in Green et al. (1987a, 1987b).

11 It is straightforward to see in which sense $g \to 1/g$ is a particular case of the $SL(2, \mathbb{Z})$ transformation.

12 For an introduction to supergravity and a general relativistic perspective on string theory, see Ortín (2004).

13 See Cecotti (2010) for an elaboration on this point. For a general discussion of foundational issues regarding two-time theories, see Weinstein (2008).

14 Interestingly enough, the idea of two times has already been discussed in the metaphysics literature in the form of the notion of hypertime, where the hypothesis of a second time is discussed to solve consistency issues in time travel scenarios (Nicholas & Smith, 2011). However, for the purposes of this paper, we will not pursue this analogy.

15 It is usually understood as a matter of convention whether or not time dimensions are associated with a minus or plus sign in the manifold's signature. The crucial condition is usually taken to be that time and space dimensions must have opposite signs in the manifold's signature. Note, however, that some, e.g. De Haro and Butterfield (forthcoming), disagree and take the distinction between time with a minus sign and time with a plus sign to be physically significant. In that case, the choice of having a signature with minus(es) assigned to the time direction(s) is usually taken to be the sensible one.

16 The way these two conditions are phrased might make it seem that (ii) depends on (i), and so that it cannot be the case that one is satisfied while the other is not. This is just an artifact of the specific formulation used in the main text, chosen for ease of exposition. An exact formulation would involve the removal of all spatiotemporally loaded language from (ii) in favor of a more neutral but linguistically cumbersome presentation in terms of manifolds and their properties.

17 See Wallace (2022) for a philosophical introduction to the treatment of general relativity as an effective field theory and some of the philosophical questions it raises.

18 We say purely formal because, as customary when dealing with path integral in quantum field theory, the integration measure is not well-defined; hence, strictly speaking, the path integral does not make sense as an integral, and is a purely formal expression used to generate expressions for quantities like scattering amplitudes within the framework of perturbation theory.

19 This point also helps clarifying an important difference between spacetime, and the compact dimensions in particular, in M-theory and in F-theory. As Cecotti, (2010, 94) remarks:

> In comparing M-theory and F-theory on the "same" elliptic Calabi-Yau CY_m we have to recall the fundamental physical difference in the role of this manifold on the two sides of the duality: the graviton propagates on the full manifold CY_m in the M-theory case, while it lives on a real codimension 2 "brane" in the F-theory. The fact that the metric does not propagate in the fiber directions, means that all distances are zero in that direction.

20 For technical background on branes in string theory, see Polchinski (1996) and Johnson (2001); the original article where branes are introduced is Polchinski (1995), while an accessible overview of the main ideas behind branes is given by Moore (2005). A celebrated application of branes to understand the non-perturbative physics behind string theory is the Vafa-Strominger microscopic calculation of the black hole entropy in string theory (Vafa, 1996); see De Haro et al. (2020); van Dongen et al. (2020) for philosophical discussion, and for the development of the AdS/CFT

correspondence (Maldacena, 1999; Witten 1998); both topics heavily rely on the physics of branes for their formulation and, in the case of AdS/CFT, for its embedding in string theory.

21 For a more complete and detailed articulation of the common core approach, and a different outlook on the ontology of dualities, see De Haro and Butterfield (forthcoming).

22 For a defense of the task of articulating the ontology of non-fundamental theories in the context of quantum field theory and in particular of the renormalization group and effective field theories, see Wallace (2006), Fraser (2018), Williams (2019).

23 Famously, general relativity is a classical theory that does not include quantum phenomena, while quantum field theory, as currently understood, is incapable of describing gravity. Indeed, reconciling these two theories is the basic problem of quantum gravity.

24 Modulo considerations regarding its actual empirical confirmation, which we put aside for the purposes of this paper while noticing that similar concerns are endemic for theories of quantum gravity.

25 Branes in string theory are usually classified according to their number of spatial dimensions so that a 10-dimensional brane like $\mathcal{X}$ is called a 9-brane since it has nine spatial dimensions.

26 Each of these fields carries with itself its supersymmetric partner, i.e., a partner field with a spin differing by $\frac{1}{2}$ compared to the one in the main text. In a supersymmetric theory like F-theory, these two fields are related by a supersymmetry transformation.

27 Plus the Higgs field, which has spin 0.

28 Non-spatiotemporal since the graviton does not propagate in all the 12 dimensions, as discussed above.

29 Cecotti (2010) goes as far as using a description of F-theory employing only the tools of complex algebraic geometry; ironically, this choice is made, according to Cecotti (2010), in the service of clarifying the *phenomenological* aspects of the theory and its relation to particle physics experiments.

30 Many thanks to Baptiste Le Bihan for insights on this point.

31 See Huggett and Wüthrich (forthcoming) for an extensive discussion of the state of the art of the debate on spacetime emergence.

32 By complex structure, we mean a linear automorphism over the tangent space of a manifold which sends I, the identity, to $-I$.

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11. Hydrodynamics on (Mini)superspace or a Non-linear Extension of Quantum Cosmology

An Effective Timeless Framework for Cosmology from Quantum Gravity

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Abstract
We outline the content and theoretical support for the proposal of “hydrodynamics on (mini)superspace” (or a non-linear extension of quantum cosmology) as an effective framework for quantum gravity in a cosmological context. The basis for the proposal is a general correspondence between hydrodynamics and cosmology, and a picture of the universe as a quantum gravity condensate. The support comes from several directions. One is from mathematical

physics: the hydrodynamics of quantum fluids can be mapped to relativistic cosmological dynamics, and both share the same conformal symmetries, which can be unraveled via geometric methods in superspace. The second is that the proposed framework is realized in quantum gravity formalisms like group field theory, in which an emergent cosmological dynamics can be extracted from the hydrodynamics of fundamental quantum simplices in a condensate phase. The same proposal can be motivated from the idea of 3rd quantization of gravity and from recent work on quantum cosmology, as an effective way of incorporating topology changing processes or cosmological inhomogeneities, respectively. A key conceptual ingredient is the relational understanding of space and time, which makes superspace the natural arena for gravitational dynamics, as opposed to the “spacetime” manifold, together with the general idea of emergent spacetime. The proposal and the results supporting it suggest an exciting dialogue between quantum gravity, the theory of quantum fluids and cosmology, as well as a new direction for analogue gravity simulations in the lab.

11.1 Motivation and Context

In this contribution we outline content, motivation and support for a new general framework for cosmology, understood as an effective description of an underlying theory of quantum spacetime and gravity, obtained upon suitable coarse graining of its constitutive structures. A first set of motivations come from modern cosmology. Cosmology has produced a basic working account (the ΛCDM model of the cosmic history to an amazing level of detail, several theoretical models for its completion and improvements, and a large array of observations to test them. A number of outstanding challenges remain, though. They include the problem of the cosmological singularity at

the beginning of our cosmic history, the choice of initial state of the universe, which affects the later evolution, the problem of transplanckian modes of cosmological perturbations in the early universe, and the dark energy (or cosmological constant) puzzle of the late (present) cosmological acceleration. Proposals of theoretical models solving one or more of these outstanding issues abound, but, in various ways, all need support or completion from a more fundamental theory of quantum gravity.

Let us give some examples. Models of the early universe are expected to provide an answer to two main questions: why do we have an approximately homogeneous, isotropic and flat universe? why an approximately scale invariant power spectrum for the cosmological perturbations? Answering these questions often ends up relying on assumptions about what happens at or close to the big bang singularity and more generally about quantum gravity physics (Barrau, 2017; Brandenberger, 2010). In inflation, one has to clarify what produces the very accelerated expansion defining it (or the origin and nature of the inflaton, if such a new field is postulated); the physics of transplanckian modes may be made relevant by the very inflationary expansion, and is not fully under control by semiclassical field theory; also, if the inflationary phase is long enough, the relevant energy scale becomes dangerously close to the Planck scale where quantum gravity effects are expected to be dominant (and again not under control in the semiclassical framework); moreover, the fact that the inflationary universe still contains a singularity suggests its incompleteness and thus the possibility that whatever quantum gravity mechanisms resolves the singularity will affect the initial state of the universe as postulated in inflationary scenario or the features of the effective field theories used in it. In bouncing cosmologies, the need for the new physics of some quantum gravity theory to corroborate or modify the

assumptions made about the bouncing phase is obvious, and the same can be said about the transition between the pre-big bang static phase and the post-big bang expanding phase in emergent universe scenarios (e.g. in string gas cosmology). In the late universe, the accelerated expansion can in principle be accounted for, at least in terms of fitting observations, by a cosmological constant, thus simply a new parameter in a classical gravitational theory (unless cosmological tensions force us to adopt a dynamical vacuum energy picture even for purely data fitting purposes); however, its value depends on quantum fluctuations of gravitational and matter fields, and on their interaction, thus requiring a more fundamental theory of quantum gravity for achieving proper explanatory power (Carroll, 2001). If dark energy is instead the result of a new type of field, then again a more fundamental theory is called for to ground it beyond the phenomenological level. If instead one opts for an account based on some modified gravity theory, this may be taken to represent a new fundamental theory of gravity, if it can be promoted to the quantum domain, or an effective classical-looking description of what is truly given by new quantum gravity effects at large scales (Brax, 2018). In turn this would imply that quantum gravity is not confined, as the effective field theory intuition would say, to the high energies/small distances domain. The latter hypothesis would be in line with emergent spacetime scenarios (Padmanabhan, 2014).

A second set of motivations comes from quantum gravity. There are many tentative quantum gravity formalisms, with different (and perspective-dependent) degrees of success and development. All of them have produced many mathematical results, conceptual insights and stimulating suggestions. They also offer many tentative indications of possible (and testable) phenomenology, including in cosmology. At the same time, they all face difficult outstanding challenges. Some of these challenges

have to do with identifying and controlling the fundamental degrees of freedom of the theory and their quantum dynamics. Others have to do with controlling (mathematically and physically) the continuum, collective or macroscopic approximation of the theory and/or the reduction to effective spacetime physics in terms of quantum fields on (dynamical) geometric backgrounds. The second challenge includes the issue of connecting to cosmological dynamics starting from the fundamental quantum gravity formalism. A common strategy is to develop toy models that incorporate ingredients from the full theory into simplified contexts tailored to cosmological physics. Quantum cosmology is a well explored example. These toy models are very useful for guessing how the full theory could deal with cosmological issues, and can even produce viable predictions of new effects. However, lacking a clear embedding in or derivation from the full theory, their insights and their predictions are only partially compelling. Connecting cosmology to quantum gravity more fully, on the other hand, requires the development of proper approximation schemes within the full theory, which are likely to involve some form of coarse graining of quantum gravity degrees of freedom.

Let us give an example of quantum gravity structures, hopefully clarifying why this should be the case. In a number of quantum gravity approaches, notably canonical loop quantum gravity, spin foam models, lattice quantum gravity (in connection variables) and tensorial group field theories, the fundamental quantum states are encoded in purely combinatorial and algebraic data: spin networks, i.e., graphs labelled by representations of Lie groups; and quantum histories of such states are expressed in the same language as spin foams, i.e., 2-complexes labelled by the same data. These combinatorial-algebraic structures can be obtained from the quantization of piecewise-flat or piecewise-degenerate geometries. In which case the

algebraic data admit a discrete geometric interpretation, and reproduce such geometries for a subclass of 'semi-classical' configurations and in specific approximations. These new 'pre-geometric' structures are then subject to a quantum dynamics, which will then also be expressed without the usual spacetime language.

This is the first general point to note: in the fundamental description of the universe and of spacetime, there may be no 'spacetime' manifold, no spatial or temporal directions, and no fields as we usually take as basic ingredients of our physical models. It is in this sense that space and time may be emergent, in quantum gravity, i.e., they may be useful and physically salient notions with associated mathematical counterparts in some approximation only (Oriti, 2014). The second general point is that, given these fundamental structures, it is even more important to distinguish two conceptually, physically and mathematically distinct types of approximations one can perform. We can call them, for simplicity, the classical and the continuum approximations, even though both will take very different concrete forms in different formalisms (especially the second may not be directly expressed as a continuum limit in the usual sense of lattice theories). The first is the one taking from the quantum description of a given set of entities (particles, fields, etc.) to a classical description of *the same entities*, restricting attention to states and dynamical regimes in which quantum effects are negligible (this is usually associated to 'taking $\hbar$ to zero'). The second corresponds to restricting attention to a collective (dynamical) regime of (many of) the original quantum entities, loosing track of their individual properties and trading them for the properties of new averaged, collective entities. This second type of approximation may entail a form of classical limit with respect to the *new* collective entity, but depend crucially on the quantum nature of the original ones.

Examples include moving from the physics of photons to that of the EM field, or from the theory of atoms to that of the quantum liquid they form upon condensation. Many examples can be exhibited to show that the two approximations/limits, classical and continuum, are physically distinct and *do not commute*, in general. We should expect the same for the fundamental quantum gravity entities and the approximations needed to recover spacetime geometry and fields from them (Oriti, 2018, 2020).

All the motivations coming from theoretical cosmology and those coming from quantum gravity, lead us to focus our attention on the problem of the emergence of cosmological dynamics from fundamental quantum gravity, as a special case of the more general problem of the emergence of continuum spacetime physics from it. We ask then what could be the effective framework that captures cosmological physics including some quantum gravity effects, obtained from the fundamental theory upon coarse graining of its microscopic degrees of freedom (which could be non-spatiotemporal in nature). The proposal we illustrate in the following is that cosmology can be connected to, in fact embedded in, quantum gravity via hydrodynamics. Specifically, we will argue that the effective framework we are looking for is given by "hydrodynamics on minisuperspace." Minisuperspace can be understood as the configuration space of possible continuum field values at a single (manifold) point, or equivalently the configuration space of homogeneous fields.

We will outline our proposed framework, hydrodynamics on (mini)superspace, in Sect. 11.2, together with its conceptual basis. Then we will offer the current evidence in support of this proposal. We will start, in Sect. 11.3 with some intriguing results in mathematical physics, showing that standard hydrodynamics of superfluids can be mapped

to relativistic cosmology, symmetries included, giving one specific example, and then indicating other known cases, on the one hand, and, on the other hand, how it is less surprising if the hydrodynamic system is understood as defined on minisuperspace. In Sect. 11.4 we will then summarize recent results in one quantum gravity formalism, i.e., tensorial group field theory (TGFT), showing how cosmological dynamics emerges exactly from its hydrodynamics sector, defined on minisuperspace. The tensorial group field theory formalism is itself a convergence of several quantum gravity formalism, suggesting that the role of hydrodynamics on minisuperspace as the relevant effective framework for quantum gravity could be more general. In further support, we will then outline two other ways in which the same framework can be derived, from a quantum gravity perspective.

Finally, in Sect. 11.5 we will point out a possible important upshot of our proposal. If correct, it may indicate a new avenue for analogue gravity simulations in condensed matter systems. Indeed, the possibility of deriving cosmological dynamics from hydrodynamics (upon proper identification of its domain with minisuperspace) may imply that, contrary to conventional wisdom, analogue gravity simulations do not need to be confined to gravitational kinematics, but gravitational dynamics could be reproduced too, symmetries included, in the hydrodynamic regime of quantum fluids, and possibly also in the lab.

11.2 Proposed Framework and Its Conceptual Basis

Let us illustrate the general aspects of our proposal, which, we argue, can provide cosmology with a mathematical

completion within quantum gravity, and novel explanatory and predictive power, while conversely providing quantum gravity with a solid effective description and testing ground.

11.2.1 General Idea

This mathematical framework we propose is given by hydrodynamics on (mini)-superspace: a non-linear dynamics for a complex function (equivalently, its modulus and phase, encoding 'fluid density' and 'velocity') on the configuration space of spacetime fields 'at a spacetime manifold point' (equivalently, homogeneous field configurations, or 'minisuperspace,' in the quantum gravity terminology).[1] In such framework, cosmological quantities (scale factors, energy densities, etc.) are hydrodynamic averages, satisfying dynamical equations derived from the hydrodynamic ones. Formally, hydrodynamics on minisuperspace can be seen as non-linear extension of quantum cosmology (Banerjee et al., 2012), i.e., the straightforward quantization of the homogeneous sector of GR. As already mentioned, quantum cosmology is a well-explored simplified framework for studying possible QG effects in cosmology, in need for a proper embedding in a complete QG formalism. In our proposal such embedding results in a non-linear (and non-local) extension, together with a radical shift in perspective.

Consider the variables of quantum cosmology.[2] These are the metric data for a homogeneous spatial geometry, i.e., the three scale factors a_i along three independent directions,[3] and their conjugate variables corresponding to extrinsic curvature information. In addition, one can have several (homogeneous) matter fields, e.g. scalar fields, and their conjugate momenta. Minisuperspace, in this case, is the space of possible values of the three scale factors and of the scalar fields, thus $\mathbb{R}^3 \times \mathbb{R}^N$, where N is the number of

scalar fields, and we consider negative values of the scale factors to encode the same geometry of the positive values but with opposite orientation. The hydrodynamic variables defined on such domain can then be combined in a complex function $\Psi : \mathbb{R}^3 \times \mathbb{R}^N \longrightarrow \mathbb{C}$. Consider the case of single matter scalar field for simplicity. The dynamical equations of hydrodynamics on minisuperspace will then take the general form:

$$\mathcal{K}\left(a_i, \partial_{a_i}, \phi, \partial_\phi\right) \Psi(a_i, \phi) \, + \, \mathcal{V}^{'}\left[\Psi\right] \, = \, \mathcal{K}\left(a_i, \partial_{a_i}, \phi, \partial_\phi\right) \Psi(a_i, \phi) + \\ + \lambda_3 \int \Psi(a_{i_1}, \phi_1) V_3(a_{i_1}, a_{i_2}, \phi, \phi_1) \Psi(a_{i_2}, \phi_2) \, + ... \, = \, 0 \tag{11.1}$$

that is, they will be integro-differential equations for Ψ (and its complex conjugate) that will be generically non-linear and non-local on minisuperspace, with the precise form of non-locality encoded in the interaction kernels V_i, where we assumed that the non-linear functional of Ψ can be expanded as a polynomial. The standard quantum cosmology formalism would correspond to a linear equation for the same Ψ, with the operator $\mathcal{K}$ given by the Hamiltonian constraint operator of a canonically quantized homogeneous sector of GR (or other classical gravitational theory). Here we are not assuming that this is the case for $\mathcal{K}$.

The new framework should be seen as a general coarse grained description of fundamental quantum gravity, not tied to a single QG formalism, in cosmological context (close to homogeneous backgrounds), incorporating (some) QG physics and inhomogeneities. We believe that it may represent a universal framework for the interface between quantum gravity and cosmology (thus, spacetime physics), incorporating a first set of quantum gravity effects, and that it could be derived from a variety of viewpoints and quantum gravity formalisms. We are going to give a variety

of evidence in support of this belief, in the following, and we will see also a number of concrete realizations.

11.2.2 Conceptual Basis: Relationalism and Emergence

Before doing so, we want to emphasize the conceptual basis of our proposal, some key perspectives that motivate, partially justify and provide a context to it.

Relationalism about spacetime—In formulating this proposal, we are embracing a relational perspective on the construction of observables in a classical and quantum gravitational context, and on spacetime physics more generally. The main lessons from the diffeomorphism invariance and background independence of classical General Relativistic physics, which we assume is to be maintained in the quantum domain, are that there exist no absolute notion of temporal or spatial direction or location or distance, and that the 'spacetime' manifold on which we routinely formulate spacetime physics has only a global role (providing a global restriction on the set of allowed geometries), i.e., that its local structures (points, directions, paths, coordinate frames, etc.) have only a practical use but no physical significance; what is physically significant are the values of dynamical fields (among which the metric field) and relations among them (Giulini, 2007).

No function of manifold points, including the metric components, or the Ricci curvature scalar, or the value of a matter scalar field at a point, is an observable, because not invariant under diffeomorphism transformations. Quantities that are global with respect to the manifold, i.e., averages of scalar functionals of fields over the whole manifold (e.g. the total 4-volume), on the other hand, are technically invariant, thus formally observables, but they are clearly not very useful for local spacetime physics. The relational

strategy, which has a long history in gravitational physics (see Tambornino, 2012; Hoehn et al., 2021 for reviews), seeks to identify space and time in relations among dynamical fields, necessarily including the metric field (to provide a notion of spacetime extension), with one set of them, e.g. suitable matter fields, used as clock and rods to parametrize (in fact *define*) the evolution and localization of the others.

A schematic and simple example of the application of such strategy, in a context of spatially homogeneous fields, starts from two quantities like the spatially homogeneous Ricci scalar $R(t)$, function (via the homogeneous metric) of some temporal coordinate t, and a matter scalar field $\phi(t)$ with the same dependence, proceeds by inverting (when possible) this dependence to obtain $t(\phi)$ and then uses it to define the diffeomorphism-invariant observable $R(\phi)$ which is interpreted as providing the time evolution of the Ricci scalar as a function of the time measured by the clock ϕ.

Ideally, spacetime physics should only be expressed in terms of such relational quantities between fields like metric, gauge fields, matter fields, for both time and space localization. This is, in general, not possible in practice. Despite its long tradition, and the fact that the relational perspective is widely accepted to be *correct* as a matter of principle, among relativists and in the quantum gravity community, manifold points, coordinates, trajectories on the manifold etc are routinely used in gravitational physics. There is nothing wrong with this, of course, if only because it clearly works well. There is also no contradiction with the relational perspective: these structures are ‘useful fictions’ that play (well) the role of physical frames in the approximation in which their actual physical properties (energy contribution and backreaction on other fields, quantum properties, etc.) are negligible.

One lesson from the above is that physics and dynamics take place, strictly speaking, on superspace, i.e., the space of field configurations, not on the spacetime manifold, which is only a (useful) auxiliary structure. We do not have, however, a formulation of General Relativity (or other gravitational theories) purely in such relational language, i.e., in terms of equations only involving field values and no manifold, save very special cases. Things become much simpler in a cosmological context, where one can adopt the assumption of (approximately) homogeneous fields.

In the homogeneous and isotropic restriction, for example, spacetime dynamics the metric only depends on the scale factor $a(t)$ and a lapse function $N(t)$ (which can fixed freely), with the scale factor encoding the universe volume (up to a constant) as $V = V_0 a^3$ (in the appropriate gauge). If the universe contains also a single free massless scalar matter field $\phi(t)$, the GR action reduces to $S = \frac{3}{8\pi G}\int dt N \left(-\frac{aV_0\dot{a}^2}{N^2} + \frac{V}{N}\frac{\dot{\phi}^2}{2N}\right)$, invariant under the 1d diffeos corresponding to reparametrizations of the time coordinate t. This should disappear from the relevant physics, which is fully captured by the diffeo-invariant relational observable $V(\chi)$ (obtained via the same 'inversion step' mentioned earlier) and by the relational evolution $\left(\frac{1}{3V}\frac{dV}{d\phi}\right)^2 = \frac{4\pi G}{3}$ involving purely relations on the configuration space (minisuperspace) $\mathbb{R}\times\mathbb{R}$ defined by all values of the pair $\{a, \chi\}$. The 'time mani-fold' $\mathbb{R}$ with coordinate t has disappeared from the picture. The above relational formulation can be generalised to inhomogeneities and cosmological perturbations (Giesel et al., 2010), and beyond, so it is not confined to toy models. To summarize: to identify spacetime with the manifold supporting fields and spacetime physics with physics on that manifold is at best an approximation of a relational formulation in terms of correlations between fields, some of which used as

reference frames. This conclusion has many conceptual and physical implications, only partially explored. From the point of view of quantum gravity, one main implication is that we do not expect to find the manifold and its structures at the fundamental level, nor in its effective description, if not introduced ad hoc for convenience.

Spacetime emergence—As already discussed, the fundamental formulation of a quantum theory of spacetime and gravity may not be the result of a straightforward quantization of some classical gravitational theory like GR, and it may involve structures other than quantized continuum fields. These may actually be (as in canonical LQG, spin foam model and group field theory) purely combinatorial and algebraic structures instead, or discrete counterparts of continuum fields, as it is the case also in simplicial quantum gravity. Continuum spacetime and fields, and the (relational) geometric observables constructed from them would be the result of their collective quantum dynamics, and be 'emergent' in the same sense in which fluid dynamics and fluid variables are emergent from the point of view of atomic physics. Notice that this applies to *all* GR structures and dynamics; this includes flat spacetime, which would be itself a highly excited, collective state of the so-called 'QG atoms' (we are thus beyond the 'emergent gravity' picture, where gravitation and curved geometries possibly emerge from flat spacetime physics). Indeed, we embrace a perspective on quantum spacetime as a peculiar (background-independent) quantum many-body system. Within it, the extraction of spacetime, and cosmology in particular, is similar to the typical problem in condensed matter theory, i.e., the extraction of macroscopic, effective physics from the atomic one. Coarse graining procedures acquire then a central role, with the GR dynamics to appear as the approximate description of the collective quantum

dynamics of many (infinite, in the idealized limit) ‘QG atoms’.

What should cosmology be, from this perspective? *Cosmology is expected to correspond to the most coarse-grained dynamics of the QG atoms*. In other words, it should be the effective dynamics of special global observables of the full theory, the result of a full coarse-grained description that restricts attention to such global quantities only. From the point of view of a quantum many-body system: this is the basic idea of a hydrodynamic regime. Thus, we may expect that cosmological dynamics will correspond to, or be extracted from, the quantum gravity hydrodynamics, and it perhaps shares several features with the hydrodynamic description of standard fluids. While this intuition needs to be substantiated by explicit derivations and more evidence in order to be compelling, the above remarks should at least clarify what is the conceptual context within which our proposal takes form. Now we move on to provide the evidence for it.

11.3 Classical Hydrodynamics/Cosmology Correspondence and Its Symmetry Basis

We will now show that standard hydrodynamics can be mapped to relativistic cosmology, symmetries included. We will give some details on a few examples only, but the correspondence is more general. In fact, already the known examples hint at a deeper level of analysis, that could (and should) substantiate and explain the correspondence on physical grounds. Here, we will confine to the mathematical level.

11.3.1 Hydro/Cosmology Map: General Idea and Example

The general idea behind the mathematical correspondence is the following. Given a certain hydrodynamic system expressed as a scalar field theory, we can identify and compute a certain set of hydrodynamic averages, or integral moments of the distribution over the (single-atom) configuration space given by the fluid field. These hydrodynamic averages satisfy dynamical equations that are equivalent to the original hydrodynamic equations. Upon a precise correspondence between the hydrodynamic averages and cosmological variables (combinations of scale factors and matter energy densities), the dynamical equations for them can be shown to match. Let us illustrate this idea with the earliest example of such correspondence, found in Lidsey (2013). Consider the (cubic) Gross-Pitaevskii equation for a 2d, cylindrically symmetric, weakly interacting BEC (assuming a large conserved particle number), in flat space, described by a mean field $\Psi(r,t)$, given by:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2\,m}\nabla^2\Psi + V(r,t)\Psi + g|\Psi|^2\Psi \tag{11.2}$$

where $V(r,t) = m\omega^2(t)r^2/2$ is the trapping potential, g is the interaction coupling among the atoms of mass m. The problem of solving this equation can be translated into solving the dynamical equations satisfied by the integral moments, representing the total particle number, width of the wave packet, radial momentum and energy of the system, respectively:

$$I_1(t) = \int d^2x|\Psi|^2 \qquad I_2(t) = \int d^2x r^2|\Psi|^2 \tag{11.3}$$

$$I_3(t) = i\int d^2xr\left[\Psi\frac{\partial\Psi*}{\partial r} - \Psi*\frac{\partial\Psi}{\partial r}\right] \quad I_4(t) = \frac{1}{2}\int d^2x\left[|\nabla\Psi|^2 + g|\Psi|^4\right] \tag{11.4}$$

with the equations being:

$$\frac{dI_1}{dt} = 0 \qquad \frac{dI_2}{dt} = I_3 \qquad \frac{dI_3}{dt} = -2\omega^2 I_2 + 4I_4 \qquad \frac{dI_4}{dt} = -\frac{1}{2}\omega^2 I_3 \tag{11.5}$$

and a conserved quantity $Q \equiv 2I_4 I_2 - \frac{I_3^2}{4}$. In turn, these equations are equivalent to the Ermakov-Pinney equation $\frac{d^2X}{dt^2} + \omega^2 X = \frac{Q}{X^3}$ for the variable $X = I_2^{1/2}$. The dynamics encoded in these equations for the hydrodynamic averages can be mapped to the cosmological dynamics encoded in the Friedmann equations for a relativistic (homogeneous and isotropic) universe with spatial curvature *k* filled with a scalar field with arbitrary potential. Let us see this map.

The Friedmann equations, and corresponding conservation law, expressed in cosmic time, look like:

$$H^2 = \frac{1}{a^2}\left(\frac{da}{d\tau}\right)^2 = \frac{2}{3}\rho - \frac{k}{a^2} \qquad \frac{d\rho}{d\tau} + 3\,H(\rho + p) = 0 \tag{11.6}$$

where the energy density and pressure of the scalar field have the expression $\rho = \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 + U(\phi)$ and $p = \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 - U(\phi)$. When written in 'laboratory time' $d/d\tau = a\,d/dt$, the Friedmann equations can be combined to give the same Ermakov-Pinney equation we had obtained for the BEC hydrodynamics: $\frac{d^2a}{dt^2} + \left(\frac{d\phi}{dt}\right)^2 a = \frac{k}{a^3}$. This means that one has a map between dynamical equations corresponding to a map between dynamical variables given by:

$$\begin{gathered} I_2 \leftrightarrow a^2 \qquad H^2 \leftrightarrow \frac{1}{4}\frac{I_3^2}{I_2} \qquad I_3 \leftrightarrow 2\frac{da}{d\tau} \\ \left(\frac{d\phi}{d\tau}\right)^2 \leftrightarrow \omega^2 I_2 \qquad I_4 \leftrightarrow \frac{\rho}{3} \qquad p \leftrightarrow I_2\omega^2 - 3I_4 \qquad Q = k \end{gathered} \tag{11.7}$$

with the matching of conservation laws:

$$\frac{dI_4}{dt} = -\frac{1}{2}\omega^2 I_3 \leftrightarrow \frac{d\rho}{d\tau} + 3\,H(\rho + p) = 0 \quad . \tag{11.8}$$

The existence of such map is intriguing enough, but in itself it may sound like a mathematical coincidence only. Still, it applies to a number of other cosmological models and hydrodynamic systems. Indeed, similar maps to different kinds of BEC hydrodynamics have been found in (D'Ambroise & Williams, 2010; Phetnora, 2010; Gumjudpai, 2009) to apply for: anisotropic Bianchi I cosmologies, Friedmann universes with cosmological constant, isotropic universes filled with matter satisfying different equations of state, inflationary universes for different inflaton potentials, universes filled with phantom-like dark energy, etc.

11.3.2 Matching Symmetries

In fact, the correspondence between BEC hydrodynamics and cosmological dynamics extends beyond the equations of motion to an exact match of symmetries. On the BEC side (Ghosh, 2002; Horvathy & Zhang, 2010; Cariglia et al., 2016), for the same kind of fluid discussed above, the symmetry we have is the 2d centrally extended Schrödinger algebra $sh(2) = (sl(2,\mathbb{R}) \times U(1)) \ltimes \left(\mathbb{R}^2 \times \mathbb{R}^2\right)$, with generators (hydrodynamic averages) and the central extension is given by the (conserved) total particle number:

$$P_i = \int \rho\partial_i\theta \quad \text{(translations)} \qquad B_i = \int (\rho x_i - tP_i) \qquad \text{(Galilean boosts)}$$

$$J = \int e^{ij}x_iP_j \quad \text{(rotations)} \qquad Q_+ = \int \left(\frac{1}{2}\rho\gamma^{ij}\partial_i\theta\partial_j\theta + |\rho|^4\right) \quad \text{(Hamiltonian)}$$

$$Q_0 = \int \left(tQ_+ - \frac{1}{2}xP\right) \quad \text{(dilatations)} \quad Q_- = 2tQ_0 - t^2Q_+ + \frac{1}{2}\int \rho x^i x_i \quad \text{(conf. transf.)}$$

What about the cosmology side of the supposed correspondence? It may seem impossible that the above

Schrödinger symmetry is present there too. The GR dynamics has a symmetry given by the 4d diffeomorphisms, which, reduced by homogeneity and isotropy, leaves only time-reparametrizations or 1d diffeos as the only remnant spacetime symmetry. Indeed, the symmetry we are looking for cannot and should not be a 'spacetime symmetry' i.e., a group of transformations acting on the 'spacetime' manifold inducing then transformations on the dynamical fields. These are redundancies, for a generally covariant theory, and the manifold itself should disappear from a more fundamental formulation of cosmological physics, as we argued above. We are instead searching for symmetries with respect to transformations acting directly on field configurations, i.e., superspace, mapping one set of field values to another, without passing via the 'spacetime' manifold. Such symmetries should exist beyond the 'spacetime' transformations, in particular beyond any diffeomorphism invariance.

Assuming that such symmetries may exist, how do we identify them? Two complementary methods are particularly aligned with the perspective we are taking here. Both methods involving 'geometrizing' the mechanical system of interest, and reformulating it as a dynamical system on the configuration space of mechanical variables (e.g. fields). We refer to Ben Achour et al. (2023) for the general theory and for the application to cosmology and gravitational dynamics. Here we only outline the basic idea and results. One method involves reformulating a general mechanical system for variables χ^a as the motion of a point particle on superspace with action:

$$S[\chi^a, \dot{\chi}^a] = c \int d\tau \left(\frac{1}{2} g_{ab}(\chi) \dot{\chi}^a \dot{\chi}^b - V(\chi) \right) \tag{11.9}$$

where the time coordinate plays now the role of affine parameter along the superspace trajectory, $g_{ab}(\chi)$ is the

supermetric with line element on superspace $ds^2 = cg_{ab}(\chi)d\chi^a d\chi^b$, and we have gauge fixed the reparametrization invariance of the affine parameter. Equivalently, we can see the same system as the free motion of a point particle on the conformal superspace, upon conformal rescaling of the supermetric, and redefinition of the affine parameter:

$$S[\chi^a, \dot{\chi}^a] = c \int d\eta \left(\frac{1}{2} G_{ab}(\chi) \dot{\chi}^a \dot{\chi}^b - 1 \right) \qquad G_{ab} = V(\chi) g_{ab} \quad . \tag{11.10}$$

One can show that classical solutions of the original mechanical system correspond to geodesics of the conformal supermetric G_{ab}, and symmetries of the mechanical systems correspond to conformal isometries of the supermetric. From the point of view of superspace alone, free mechanical systems will correspond to null geodesics. For general potential, one has instead geodesics for positive mass, as it can be seen by restoring the reparametrization invariance and computing the associated constraint equation.

A second method is an immediate extension of the first, allowing to treat even interacting mechanical systems as free motion and to study efficiently also 'time-dependent' symmetries, i.e., transformations that, while defined on superspace, depend in their definition on the 'time coordinate.' It relies on the Eisenhart-Duval lift, an extended superspace augmented by two null directions, one of which corresponding to the original time coordinate, with line element $ds^2_{ED} = 2dudw - 2V(\chi)du^2 + g_{ab}(\chi)d\chi^a d\chi^b$. Given the reliance on the time coordinate, this method does not account automatically for reparametrization invariance, which has to be imposed separately. The new point particle dynamics is then given by the action:

$$S[\chi^a, \dot{\chi}^a, u, w] = \int d\lambda \left(\dot{u}\dot{w} - V(\chi)\dot{u}^2 + \frac{1}{2} g_{ab}(\chi) \dot{\chi}^a \dot{\chi}^b \right) . \tag{11.11}$$

Now, classical solutions of the original mechanical system correspond to null geodesics of the ED-extended supermetric, and its symmetries (now including time-dependent transformations) correspond to its conformal isometries. Given these two general methods, one can investigate the hidden symmetries of cosmological and other gravitational systems (Ben Achour et al., 2023; Ben Achour & Livine, 2020). For the Friedmann dynamics of a homogeneous and isotropic universe filled with a free massless scalar field, one identifies symmetry transformations in field space corresponding to all the ones found in the BEC hydrodynamics, i.e., the counterpart of translations (now in field space/superspace), Galilean boosts, conformal transformations, dilatation and rotations, and forming again the 2d centrally extended Schrödinger algebra $sh(2) = (sl(2,\mathbb{R}) \times U(1)) \ltimes \left(\mathbb{R}^2 \times \mathbb{R}^2\right)$.

Interestingly, the central extension is now given by $n = \frac{2V_0}{L_p^3}$, where V_0 is the volume of the fiducial cell of the cosmological model (over which one restricts spatial integrations, and thus playing the role of iR cutoff) and L_P is the Planck length, playing the role of UV cutoff; in other words, it is given by the total (conserved) number of Planck-sized cells constituting the (homogeneous) universe with Friedmann dynamics. The explicit form of the field transformations and conserved charges/generators can be found in Ben Achour et al. (2023). For example, the combination of translations and Galilean boosts can be written as:

$$\begin{aligned} z \longrightarrow \tilde{z}(\tilde{\tau}) &= z(\tau) + \frac{\xi(\tau)}{2} e^{\pm L_p}\phi/2 \\ \phi \longrightarrow \tilde{\phi}(\tilde{\tau}) &= \phi(\tau) - \frac{\pm\xi(\tau)}{L_p z} e^{\pm L_p}\phi/2 \end{aligned} \tag{11.12}$$

where $z = a^{3/2}$ and ξ is an arbitrary function of τ, and the translation generators are:

$$P_{\pm} = e^{\mp L_p \phi/2} \left[\frac{p}{2} \pm \frac{\pi}{L_p z} \right] \tag{11.13}$$

with $p = -2nL_p N^{-1} \dot{z}$ is the conjugate variable to z for generic lapse N, and $\pi = -\frac{n}{2} L_p^3 N^{-1} z^2 \dot{\phi}$ is the conjugate momentum of ϕ.

A crucial point is that there is thus a precise unique correspondence between the conserved charges of the cosmological dynamics and the conserved charges of the hydrodynamic system; if the associated symmetry algebra is large enough, this allows to reconstruct fully one dynamical system from the other, and viceversa (if not, then the correspondence still helps the reconstruction of some essential elements of the two). For example, the reconstruction is possible for the Friedmann/cubic BEC case, and one can re-derive the Lidsey correspondence entirely from the perspective of the Eisenhart-Duval lift.[4]

Despite the matching of symmetries too, that we have shown for the simplest FRW case, one could still see the hydrodynamics/cosmology correspondence as a mathematical coincidence only. We argue that there is more to it. Here we simply point out that also the symmetry-matching analysis can be generalised to a number of other gravitational systems, including Friedmann cosmologies with various matter scalar field potentials (Ben Achour et al., 2023), Bianchi I cosmology,[5] and Schwarzchild-(A)deSitter black holes (Ben Achour et al., 2023). In the next Section, we discuss how the correspondence can be motivated and, in fact, derived from a more fundamental quantum gravity starting point, from a variety of perspectives.

11.3.3 Making Sense of the Correspondence

Before moving to quantum gravity considerations, let us stress how our proposal of hydrodynamics on superspace makes this hydrodynamics/cosmology correspondence intuitively reasonable. The correspondence appears natural, at least at the mathematical level, if one understands the BEC mean field Ψ to be defined on (mini)superspace, i.e., as a function of the cosmological variables (a, ϕ), rather than 'spacetime coordinates' (r, t). This is not immediate for the original Lidsey map, because it relies on the time coordinate also on the cosmological side of the map. However, one gains an immediate intuition for the cosmological significance of the various elements of the hydrodynamic system when this is defined on minisuperspace and the cosmological quantities are defined in relational terms.

In fact, based on the symmetries, one can put in correspondence a 2nd order 'relativistic version' of the BEC hydrodynamics with a purely relational formulation of cosmological dynamics, with a free massless scalar field used as a relational clock and no role for the temporal coordinate, with the first being defined on superspace only, while the Lidsey's map is obtained when lifting the domain of the hydrodynamics to the Eisenhart-Duval lift, including the temporal coordinate as a variable.[6] In such rewriting, Lidsey's averages look like quantum cosmology averages for a mean field defined on an extended minisuperspace including the chosen time coordinate as a variable, i.e., $\Psi(a, \phi; t)$ and then foliated with respect to it. For example:

$$
\begin{aligned}
I_2(t) &= \int da d\phi \, a^2 |\Psi(a, \phi; t)|^2 = \langle a^2 \rangle_t \longleftrightarrow a^2(t) \\
I_3(t) &= \int da d\phi \, a \left[\Psi(a, \phi; t) \partial_a \Psi^*(a, \phi; t) - \Psi^*(a, \phi; t) \partial_a \Psi(a, \phi; t) \right] = \\
&= \langle a \pi_a \rangle_t = \langle aH \rangle_t \longleftrightarrow (aH)(t)
\end{aligned}
\tag{11.14}
$$

Instead, in a purely relational context, one has a mean field $\Psi(a,\phi)$ defined on minisuperspace and integral moments, i.e., hydrodynamic averages, would look like standard quantum cosmology averages giving relational cosmological observables, with respect to the foliation (of minisuperspace) defined by the choice of relational clock given, say, by the scalar field. For example,

$$\begin{aligned} I_2(\phi) &= \int da\, a^2 |\Psi(a,\phi)|^2 = \langle a^2 \rangle_\phi \longleftrightarrow a^2(\phi) \\ I_3(\phi) &= \int da\, a \left[\Psi(a,\phi)\partial_a \Psi^*(a,\phi) - \Psi^*(a,\phi)\partial_a \Psi(a,\phi)\right] = \\ &= \langle a\pi_a \rangle_\phi = \langle aH \rangle_\phi \longleftrightarrow (aH)(\phi) \end{aligned} \tag{11.15}$$

The same construction can be given for all the symmetry generators and other relevant observables.[7] Following the above intuition, the maps between hydrodynamics and cosmology, discussed in this section give further motivation to our proposal.

11.4 Quantum Gravity Support

The correspondence between classical cosmology and hydrodynamics, outlined above, even if shown to be very general, would not in itself vindicate our claim that hydrodynamics on superspace can represent a general effective framework for cosmology, encoding some quantum gravity physics. In order to do so, it is necessary to show a direct, if approximate, connection between this effective framework and fundamental quantum gravity formalisms, and possibly a concrete derivation of the the first from the second. It would be even better to show that there are many routes for such derivation, and several quantum gravity formalisms leading to the same effective framework, thus some degree of universality. This would make the theoretical aspects of the framework more

compelling, but also any eventual observational consequence in cosmological physics more robust. This is what we do in this section: we show how cosmological dynamics emerges from quantum gravity in its hydrodynamics sector, defined on superspace, from a variety of perspectives, but with different degrees of detail and robustness.

11.4.1 GFT Condensate Cosmology

The derivation is rather complete and detailed in one specific quantum gravity formalism, namely tensorial group field theories, specifically the more quantum geometric models of (Krajewski, 2011; Oriti, 2012) simply referred to as 'group field theories.' We first introduce the main elements of the formalism, then review very briefly how hydrodynamics on superspace and then cosmology emerge from it, and why this suggests that similar derivations are possible in related quantum gravity approaches. For more details on the derivation as well as on the many results of TGFT cosmology, we refer to introductions and reviews of the vast literature (Gielen & Sindoni, 2016; Oriti, 2017; Pithis & Sakellariadou, 2019). See also Oriti (2021) for an extended discussion of how TGFT cosmology provides a concrete example of a more general template for the emergence of spacetime in quantum gravity.

These quantum geometric models describe the quantum structure of spacetime in terms of quantized tetrahedra, with algebraic data encoding their discrete geometry. The initial Hilbert space of an individual tetrahedron is $\mathcal{H} = L^2\left(G^4; d\mu_{Haar}\right)$, on which one imposes appropriate 'geometricity' restrictions (which can be imposed at the level of dynamics). The Lie group G is the Lorentz group $SO(3, 1)$ (or its double cover $SL(2, \mathbb{C})$) or its rotation subgroup $SU(2)$, $d\mu_{Haar}$ is the Haar measure. Quantum states of geometry/spacetime are then functions of group

elements associated to the triangles of each tetrahedron. An equivalent representation is in terms of (unitary, irreducible) group representations associated to the same triangles; thus, one can represent the same tetrahedron as a spin network vertex, i.e., a vertex with four outgoing open links labeled by group representations.

One can then define a Fock space for arbitrary numbers of tetrahedra: $\mathcal{F}(\mathcal{H}) = \bigoplus_{V=0}^{\infty} sym\left\{\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \cdots \otimes \mathcal{H}^{(V)}\right\}$, and introduce field operators creating or annhiliating the quanta of this Fock space, i.e., quantum tetrahedra, moving to a field-theoretic language. In this Fock space, quantum states associated to extended simplicial complexes formed by gluing tetrahedra across shared boundary triangles correspond to maximally-entangled states across the quantum degrees of freedom associated to the same triangle in two glued tetrahedra. Such states correspond to the spin network states associated to closed graphs in canonical loop quantum gravity and spin foam models (Giesel & Sahlmann, 2011; Perez, 2013). So TGFT states can encode discrete (piecewise-flat) geometries, at least for special states and in a semiclassical approximation. This discrete geometric intuition guides model building and the analysis of quantum dynamics.

However, the list of 'geometric or spatiotemporal pathologies' that generic quantum states for arbitrary collections of TGFT building blocks can possess indicates the gap with respect to the usual description of spacetime in terms of fields (including a metric field). This justifies referring to this level of description as non-geometric and not spatiotemporal, and TGFT cosmology as an example of spacetime emergence. A partition function for the TGFT 'atoms,' i.e., the quantized tetrahedra, will in general take the following field-theoretic form:

$$Z = \int \mathcal{D}\varphi \mathcal{D}\varphi^* \, e^{-S_\lambda(\varphi,\varphi^*)} \quad , \tag{11.16}$$

for an action $S_\lambda(\varphi, \varphi^*) = K + U + U*$ function of the field (and complex conjugate) and some coupling constant(s) weighting an interaction term U given by a polynomial in the fields, in addition to the quadratic kinetic term K. The pairing of their (group) arguments is, in general, non-local (they are not simply identified at interactions). This form of the partition function can be motivated, when not derived, from a variety of viewpoints (see Oriti, 2021 and references therein). A key fact, providing a useful guideline, is that it can be seen as the generating function, in its perturbative expansion, for a sum over simplicial complexes (which are dual to the TGFT Feynman diagrams) each weighted by a discrete gravity path integral and incorporating also a sum over discrete topologies. For quantum geometric models, the lattice path integral associated to each Feynman diagram of the TGFT is highly non-trivial and formulated in algebraic discrete geometric variables, and in fact can be expressed equivalently as a spin foam model. Thus such TGFTs encode the continuum limit of lattice gravity path integrals and spin foam models, as well as the quantum dynamics of spin networks (quantum states of geometry in canonical loop quantum gravity), so that one can infer the appropriate TGFT action and partition function from knowledge of the lattice gravity or spin foam model they generate (and viceversa).

This basic guideline can be used also to define TGFT models for quantum gravity coupled to matter. The strategy is to define a TGFT field and action in such a way that, in perturbative expansion, one obtains Feynman amplitudes with the form of discrete path integrals for gravity coupled to scalar fields, on the lattice dual to the Feynman diagram. This requires an extension of the domain of the TGFT field to include extra components corresponding to the possible values of the matter field, e.g. to $\varphi(g_I, \chi)$ for a single scalar field χ, and a suitable coupling of these new data with the

quantum geometric ones and across TGFT fields in the action.

The simplest approximation of the full quantum effective action, thus of the full continuum limit of lattice gravity path integrals and spin foam models, is the TGFT mean field hydrodynamics. It corresponds to the saddle point evaluation of the full TGFT path integral, and to approximating the full quantum effective action with the classical TGFT action $\Gamma[\phi] \simeq S(\phi)$. From the point of view of the QG atoms, i.e., the TGFT quanta, this amounts to working with highly quantum states: coherent states $|\Psi\rangle$ whose expression in terms of TGFT Fock excitations is:

$$|\Psi\rangle = \exp(\widehat{\Psi})|0\rangle \qquad \widehat{\Psi} = \int d[g]d\chi \Psi(g_I;\chi)\widehat{\varphi}^{\dagger}(g_I;\chi), \qquad (11.17)$$

where $|0\rangle$ is the Fock vacuum (no QG atoms at all), and the exponential operator can be expanded to give an infinite superposition of states with increasing number of tetrahedra or spin network vertices.

This is the quantum gravity counterpart of the Gross-Pitaevskii approximation in the hydrodynamics of quantum liquids. The task is to obtain, from such TGFT hydrodynamics, an effective dynamics expressed in spatiotemporal terms, in the sense of quantum GR, i.e., an effective dynamics of geometric quantities as those constructed out of continuum fields including the metric. When focusing on the TGFT hydrodynamics, we are assuming: (a) that the relevant phase of the QG system is a *condensate phase* of QG entities, guided by the intuition of the universe as a kind of quantum fluid; (b) that the relevant physical information is captured by the condensate wavefunction, i.e., the mean field Ψ; (c) that a Gaussian, weakly interacting regime is already good enough to unravel interesting spacetime/gravitational physics.

The guess is that TGFT hydrodynamics will correspond to cosmological dynamics, for the heuristic reasons anticipated in the previous general discussion. Another reason follows from noticing that the TGFT mean field has the same domain of a wavefunction of a single 'GFT atom,' i.e., an individual 3-simplex, encoding its (usually spacelike) quantum geometry at a point, i.e., the type of data that would suffice to describe cosmological dynamics.

The guess is supported by a general fact, which applies to any TGFT model in which the domain $\mathcal{D}$ of the mean field $\Psi(\mathcal{D})$ (and of the fundamental field $\varphi(\mathcal{D})$) is understood as the space of geometries of a single (spacelike) 3-simplex (or conjugate extrinsic geometry), plus additional matter data. It can be shown (Gielen, 2014; Jercher et al., 2022) that such domain $\mathcal{D}$ is diffeomorphic to the space of metrics (or conjugate extrinsic curvatures) at a point in a 3d (spacelike) hypersurface, plus matter field values at the same point, which in turn is diffeomorphic to the minisuperspace of continuum homogeneous 3-geometries (or conjugate homogeneous extrinsic data), plus homogeneous matter fields. Therefore, the TGFT condensate wavefunction $\Psi(\mathcal{D})$ can be understood as a wavefunction on minisuperspace, as in quantum cosmology. The other general result is that this wavefunction satisfies non-linear dynamical equations, not the linear ones of quantum cosmology (i.e., the Wheeler-DeWitt equation restricted to wavefunctions on minisuperpsace). These are the quantum equations of motion derived from the quantum effective action or, in our approximation, the classical equations of motion of the chosen TGFT model (the quantum gravity analogue of a Gross-Pitaevskii hydrodynamic equation for a quantum fluid):

$$\int [dg'] d\chi' \mathcal{K}\left([g],[g'];\chi,\chi'\right)\, \Psi(g',\chi') \,+\, \lambda \frac{\delta}{\delta \varphi^*} \mathcal{V}(\varphi,\varphi^*)_{|_{\varphi=\Psi}} = 0, \quad (11.18)$$

with analogous equation for the conjugate TGFT field. These equations are in general also non-local on minisuperspace, due to the non-local nature of the TGFT interactions. We obtain, then, a non-linear and non-local extension of a quantum cosmological equation, for the condensate wavefunction, encoding an infinity of quantum gravity degrees of freedom in a coarse grained, collective manner. It is an explicit and quite general realization of our proposed framework of hydrodynamics on (mini)superspace, derived at an effective coarse-grained level from a full quantum gravity formalism.

The next step is then to start from the hydrodynamic equations and extract from them dynamical equations for suitable geometric observables with a clear cosmological meaning. Of course, different TGFT models will give hydrodynamic equations that differ in their detailed form, and thus different cosmological dynamics. We should also expect, however, some degree of universality across different models, since we are working an effective coarse-grained level. This is confirmed by the analysis of specific TGFT models. Here, we report a sketch of the derivation of cosmological dynamics, highlighting the main steps without model-dependent details, and then summarize some interesting results obtained in specific cases.

A major simplification occurs when restricting attention to isotropic configurations, thus to functions of a single group representation label, once expanded in irreps of the group, and on the scalar field values only: $\Psi^j(\chi)$, corresponding to the fact that one single metric degree of freedom, e.g. the universe volume (or the scale factor), is relevant. A second simplification often applied is to neglect the TGFT interactions, i.e., the non-linearities of the equations, since consistency with lattice path integral (and spin foam) guidelines requires these interactions to be very

weak. However, the effect of the interactions has also been studied, with interesting results.

A key step is then to turn either the hydrodynamic equations directly or the ensuing equations for observables into relational form, by using one of the dynamical variables in the domain of the mean field as a relational clock. Often this role is played by the scalar field χ, and there are several strategies to take this step, in the TGFT cosmology literature. One is to deparametrize the TGFT model from the start, i.e., at the level of the TGFT action, which results in a different (clock-dependent) Fock space than the one we gave above, different (clock-dependent) TGFT operators and observables, and hydrodynamic equations which are already written as temporal evolution equations, different from the general form we gave above. Here, we follow another route, and introduce instead an approximate, state-dependent relational framework.

We consider condensate states that are 'semiclassical enough' with respect to the chosen clock variable, i.e., the scalar field:

$$\Psi_\epsilon(j;\chi) \equiv \eta_\epsilon(j;\chi-\chi_0;\pi_0)\tilde{\Psi}(j;\chi), \tag{11.19}$$

where η is a function (e.g. Gaussian) peaked around the χ_0 value of the clock variable χ, with width given by $\epsilon \ll 1$, and depending on a second parameter π_0 governing the fluctuations in the conjugate variable to χ (related to the momentum of the scalar field, whose fluctuations are small if $\pi_0^2\epsilon \gg 1$). The TGFT hydrodynamics equations are then well-approximated by equations for $\tilde{\Psi}$ involving 'time' derivatives with respect to χ_0, where in fact any higher order derivative (in χ_0) possibly present in the original kinetic term can be neglected with respect to the second (and first) order ones:

$$\tilde{\Psi}_j''(\chi_0) + A_j\tilde{\Psi}_j'(\chi_0) - B_j\,\tilde{\Psi}_j(\chi_0) + \mathcal{V}[\tilde{\Psi}] = 0, \tag{11.20}$$

where the coefficients A_j and B_j and the interaction functional $\mathcal{V}$ are, in general, function of both the parameters of the given TGFT model (the specific choice of TGFT action) and of the state (11.19). One then needs to compute geometric observables, and specifically relational observables whose 'temporal' localization is defined with respect to the scalar field clock χ_0. These are expectation values of fundamental TGFT operators acting on the TGFT Fock space with a clear geometric interpretation (relying on the discrete gravity picture behind the TGFT model), evaluated on the states (11.19), and thus well approximated by the value the condensate wavefunction takes at $\chi = \chi_0$. Relevant ones are the occupation number:

$$N(\chi_0) \equiv \langle \widehat{N} \rangle_{\Psi;\chi_0,\pi_0} = \sum_j \rho_j^2(\chi_0), \tag{11.21}$$

the universe volume (constructed from the 1st quantized volume operator for quantized tetrahedra, with eigenvalues V_j:

$$V(\chi_0) \equiv \langle \widehat{V} \rangle_{\Psi;\chi_0,\pi_0} = \sum_j V_j \rho_j^2(\chi_0), \tag{11.22}$$

the clock (scalar field) value:

$$\frac{\langle \widehat{\chi} \rangle_{\Psi;\chi_0,\pi_0}}{N(\chi_0)} \simeq \chi_0, \tag{11.23}$$

and the scalar field momentum $\langle \widehat{\Pi} \rangle_{\Psi;\chi_0,\pi_0}$. All these quantities, which will represent the cosmological variables of the resulting cosmological dynamics, are hydrodynamic averages computed for a hydrodynamics system defined on (mini)superspace. This is exactly in line (despite differences in the actual derivation) with the hydrodynamics/cosmology maps discussed in Sect. 11.3. Using the hydrodynamic equation (11.20), we then obtain the equations governing the relational evolution of the

universe volume as a function of the energy density (defined via the above expectation values) of the scalar field (and of any other matter field we could have included in the model:

$$\left(\frac{V'}{3V}\right)^2 = F\left(V, \rho, \theta, A_j, B_j, \mathcal{V}, \pi_0, \epsilon\right) \qquad \frac{V''}{V} = G\left(V, \rho, \theta, A_j, B_j, \mathcal{V}, \pi_0, \epsilon\right), \tag{11.24}$$

where of course the functionals F and G are again model- and state-dependent. These are the *generalised (quantum-corrected) Friedmann equations in relational time* for a given TGFT quantum gravity model, for the emergent spacetime in the homogeneous case, captured by the volume only, in the presence of the chosen simple matter content). The whole procedure can be generalised to different matter content. In particular, it can be generalised to deal with inhomogeneous universes, by introducing suitable matter frames (e.g. four free massless scalar fields) to localize observables in both space and time, in line with the relational strategy (Marchetti & Oriti, 2022).

Many results obtained in TGFT cosmology over the last 10 years include the recovering of the classical GR dynamics for large volumes, the effective dynamics of cosmological perturbations in standard field theory language, the resolution of the cosmological singularity via a very robust bouncing mechanism, several mechanisms for producing an inflationary phase in the early universe or a (phantom-like) dark energy in the late universe, both of pure quantum gravity origin (without ad hoc matter fields), anisotropies, and more. We do not discuss them. Our summary only intended to show how the framework of hydrodynamics on superspace, as an effective way of encoding cosmological dynamics, can in fact be derived from first principles in a candidate fundamental quantum gravity formalism. The viability, as well as the potential

embedding within full quantum gravity, of the proposed framework should be now clear.

The question that remains is to what extent hydrodynamics on superspace can be seen as a universal description of a cosmological regime, shared across many different quantum gravity approaches. Now we want to address at least partially this question, with two more examples of quantum gravity-related settings which also lead to the same framework. Before doing so, we stress that, in fact, already the derivation of hydrodynamics on superspace (and cosmology from it) from TGFTs suggests that it may represent a universal effective framework shared across different quantum gravity approaches. The reason is that the TGFT framework itself is a crossroad of different quantum gravity formalisms. TGFT models can be seen as an alternative definition of simplicial quantum gravity path integrals (involving a sum over triangulations) and, for quantum geometric models, as a completion of spin foam models for quantum gravity and a 2nd quantized reformulation of canonical loop quantum gravity. While the TGFT language may be particularly convenient for obtaining an effective coarse grained formulation of the quantum dynamics of spin networks, or of lattice gravity path integrals, it can also be taken to be simply an indication that such coarse grained formulation, even when achieved by other methods, will also give hydrodynamics on superspace as a result.[8]

11.4.2 3rd Quantization and Effective Topology Change

Let us now look at a different, albeit related, perspective on quantum gravity, not tied to discrete structures. Motivated mainly by difficulties in defining a positive definite inner product in canonical quantum gravity (due to the mixed signature of the supermetric) and by the will to incorporate

topology change, the 3rd quantization idea was proposed over thirty years ago (Giddings & Strominger, 1989). The idea has surfaced back recently, due to the renewed interested in wormhole production in quantum gravity (Giddings & Turiaci, 2020). It has never gained too much traction, mostly because making mathematical sense of it is even more challenging than for the gravitational path integral (or canonical quantization) itself. It has a great conceptual appeal, however, and one can also see TGFTs (and the earlier matrix and tensor models) as a concrete realization of it in a discrete gravity context (Gielen & Oriti, 2012). The basic idea is to define a 'quantum field theory of universes' by promoting the canonical gravity wavefunction to a field $\Phi[{}^3g]$ living (like the canonical wavefunction) on superspace, i.e., the space of 3-geometries for given spatial topology (we assume that of a 3-sphere). The dynamics of this superspace field (assumed real) is encoded in the action:

$$S = -\frac{1}{2}\int \mathcal{D}^3 g\, \Phi \Box \Phi \; + \; V[\Phi] = -\frac{1}{2}\int \mathcal{D}^3 g\, \Phi \Box \Phi \; + \; \lambda_3 V_3[\Phi] + ... \quad (11.25)$$

where the kinetic kernel is given by the Wheeler-DeWitt Hamiltonian constraint of canonical gravity, and the interaction terms encode topology changing processes. For example, V_3 governs the splitting of a universe into two with the same topology, i.e., a 'trousers' process or its inverse (merging of two universes into one), with a form like:

$$V_3 = \int \mathcal{D}^3 g_1 \mathcal{D}^3 g_2 \mathcal{D}^3 g_3\, \Phi[{}^3g_1]\Phi[{}^3g_2]\Phi[{}^3g_3]\delta({}^3g_2, {}^3g_1^-)\delta({}^3g_3, {}^3g_1^+) \quad (11.26)$$

where the kernel gives matching conditions at the topological junctions across universes. The classical equations of motion of this theory, $\frac{\delta S}{\delta \Phi} == \Phi = V'[\Phi] = 0$ would then be by definition dynamical equations for a wavefunction on superspace and a non-linear and in

general non-local extension of the Wheeler-DeWitt equation of canonical quantum gravity. They will represent the simplest (mean field) approximation of the full quantum equations of motion of the theory, obtained by variation of the quantum effective action $\frac{\delta\Gamma[\Phi_B]}{\delta\Phi_B} = 0$, which will be the fully quantum-corrected non-linear (and non-local) extension of the Wheeler-DeWitt equation, including the effects of topology change.

The quantum field theory defined by the action (11.25) would have a partition function which, in a perturbative expansion, would give a sum over Feynman 'diagrams' dual to manifolds of higher an higher genus, obtained connecting a propagator corresponding to a cylindrical, globally hyperbolic topology (with boundary 3-manifolds being 3-spheres, with interaction vertices corresponding to trousers topologies (assuming we only include the interaction V_3 of both orientations:

$$Z_{\lambda_3} = \int \mathcal{D}\Phi[^3\, g]\, e^{-S[\Phi]} = \sum_{\mathcal{M}} \mathcal{A}[\mathcal{M}]. \tag{11.27}$$

Each perturbative process would be weighted by a quantum gravity path integral on the given topology and involving the classical gravity action corresponding to the Wheeler-DeWitt operator $\Box$, e.g. the Einstein-Hilbert action of GR:

$$\mathcal{A}[\mathcal{M}] = \int_{\{g|\mathcal{M}\}} \mathcal{D}g\, e^{iS^{EH}_{\mathcal{M}}(g)} \tag{11.28}$$

Obviously, the theory so formulated in entirely formal and faces enormous mathematical difficulties. Already the functional measure of the gravitational path integral, entering the perturbative amplitudes as well as the action, is a major obstacle (properly defining it would amount to solving a major difficulty of canonical and covariant quantum gravity). It is no surprise, then, that achievements

in this framework have been confined to conceptual insights, formal computations suggesting possible new physics, and semiclassical or otherwise approximate results. We notice once more, however, that matrix models for 2d gravity, and tensor models and TGFTs in higher dimensions, can be seen as a more rigorous formulation in a discrete context, but in the very same spirit, and allowed much more progress (Gielen & Oriti, 2012). Among the simplifications that one can adopt to make the 3rd quantization framework more rigorous, there is also the minisuperspace reduction to homogeneous geometries and matter fields (Klebanov & Susskind, 1989). This reduces the framework to that of an ordinary scalar field theory on minisuperspace. In the case of isotropic universes governed by GR at the classical level and filled with a single matter scalar field, one has a 2d minisuperspace with coordinates $\{a, \phi\}$, and an action:

$$\begin{aligned} S =& \frac{1}{2}\int dad\phi \left[\Phi\, \partial_a^2\Phi - \frac{1}{a^2}\Phi\, \partial_\phi^2\Phi + (a^2 - a^4 - a^4U(\phi)\Phi^2\right] \\ &+\frac{\lambda_3}{2}\int dada^{'}da^{''}d\phi d\phi^{'}d\phi^{''}\Phi(a,\phi)\Phi(a^{'},\phi^{'})\Phi(a^{''},\phi^{''})\mathcal{V}\left(a,\phi;a^{'},\phi^{'};a^{''},\phi^{''}\right) \end{aligned} \tag{11.29}$$

This is exactly the sort of hydrodynamics on minisuperspace we have discussed in the previous sections, and the object of our proposal. Indeed, the corresponding equations of motion give a non-linear (and non-local) extension of the dynamics of quantum cosmology. Here, the non-linearities would encode, at an effective level, the physics of topology changing processes, in a language that only refers to a single, homogeneous universe. It should be clear that the 3rd quantization idea does not depend on the specific formulation of classical or quantum gravity/cosmology one has at hand. Indeed, the minisuperspace version of 3rd quantization has been proposed and analysed by Calcagni et al. (2012) having as starting point loop quantum cosmology, based on

connection and tetrad variables. In the variables used in the loop quantum cosmology framework, in particular ν labelling the eigenvalues of the volume operator, one has an action:

$$\begin{aligned} S_{\rm i}[\Phi] =& \frac{1}{2}\sum_{\nu}\int d\phi\ \Psi(\nu,\phi)\hat{\mathcal{K}}\Phi(\nu,\phi) + \sum_{j=2}^{n}\frac{\lambda_j}{j!}\times \\ & \sum_{\nu_1\dots\nu_j}\int d\phi_1\dots d\phi_j\ f_j(\nu_i,\phi_i)\prod_{k=1}^{j}\Phi(\nu_k,\phi_k)\,, \end{aligned} \tag{11.30}$$

for generic interaction kernels and a kinetic operator being the Hamiltonian constraint operator in (isotropic) loop quantum cosmology, given by a difference operator

$$\hat{\mathcal{K}}\,\Phi(\nu,\phi) := -B(\nu)\left(\Theta + \partial_\phi^2\right)\Phi(\nu,\phi) = 0\,, \tag{11.31}$$

$$\begin{aligned} -B(\nu)\Theta\psi(\nu,\phi) :=\ & A(\nu)\psi(\nu+\nu_0,\phi) + C(\nu)\psi(\nu,\phi) \\ & + D(\nu)\psi(\nu-\nu_0,\phi)\,. \end{aligned} \tag{11.32}$$

Here A, B, C, and D are functions depending on the details of the quantization scheme and the choice of lapse function), and ν_0 is an elementary length unit, usually defined by the square root of the area gap (proportional to the Planck length). The non-linearities have again the interpretation of encoding topology changing processes, in a 3rd quantization perspective. Different specific choices and conditions one may impose on such interactions are discussed in Calcagni et al. (2012).

In fact, another interpretation is possible for the same interactions, following which one would make different choices for the interaction kernels, with the same general form of the dynamics. Instead of a field theory of interacting homogeneous universes, one could interpret it as a field theory of interacting homogeneous *patches* of a single inhomogeneous universe. The interactions would then govern processes in which such patches merge or

split, giving an effective way of dealing with cosmological inhomoegeneities. This option is also discussed in some detail in Calcagni et al. (2012). It is also the perspective taken in following another route that leads to hydrodynamics on minisuperspace and non-linear quantum cosmology, which we now discuss.

11.4.3 Separate Universe and Effective Inhomogeneities

In Bojowald et al. (2012), the starting point is the separate-universe approach to cosmological perturbations. In this approach (Wands et al., 2000), the universe is subdivided into homogeneous regions with different geometry. Whether this is a good or too coarse description of an inhomogeneous universe depends on its precise geometry. One expects that for strong curvature and small wavelengths of the cosmological perturbations, one has to use homogeneous patches with very small volume, and thus very many of them (for given total 3-volume of the universe), in order to have a good approximation at both kinematical and dynamical level. For low curvature and long wavelengths of cosmological perturbations, large homogeneous patches would suffice.

The relative differences in cosmological variables, e.g. 3-volume or scale factors, would then measure the degree of inhomogeneity across patches. In the limiting case of very strong curvature, e.g. close to the cosmological singularity, one would have a (infinite) number of extremely small homogeneous patches, one per spatial manifold point in the actual limit. Each patch would evolve according to the dynamics of homogeneous cosmology. The BKL conjecture would then state that the individual patch dynamics for all patches is actually enough to control the full universe dynamics, i.e., that spatial gradients (relations across patches) are in fact negligible.

Let us take this starting point seriously, but confine ourselves to the less extreme situation of many homogeneous patches of finite size, individually governed by a homogeneous dynamics but with non-negligible interactions among them. The analysis of Bojowald et al. (2012) aimed at extracting an effective coarse grained dynamical framework for such universe, following the example of BECs, and in the form of a hydrodynamics of homogeneous patches. We will see that this turns out to be a hydrodynamics on minisuperspace and a non-linear extension of quantum cosmology. We sketch here the main steps of the derivation, and refer to Bojowald et al. (2012) for more details.

We split the spatial manifold $\sum$ into several homogeneous and isotropic regions: $\Sigma = \bigcup_{i,j,k=1}^{N^{1/3}} \mathcal{S}_{i,j,k}$, where the notation suggest a cubic lattice but we need to make no assumption about the precise topology of the decomposition. Because of homogeneity and isotropy, for each region it is enough to consider only the volume degree of freedom $V_{i,j,k}$, such that $V = \sum_{i,j,k} V_{i,j,k}$. For simplicity, we neglect any matter content of the universe, in the following. The dynamics is encoded in the Hamiltonian constraint of GR, which is adapted (discretized) to the regions as:

$$H_{disc} = -6\pi G V \left(\Pi_V^2 + \frac{1}{N} \sum_{i,j,k} \Pi_{i,j,k}^2 + ... \right) \qquad (11.33)$$
$$\Pi_V = -\frac{1}{12\pi G} \frac{\dot{V}}{V} \qquad \Pi_{i,j,k} = -\frac{1}{12\pi G} \left(\frac{N\dot{V}_{i,j,k}}{V} \right) \quad .$$

where we have neglected higher derivatives terms, thus assuming inhomogeneities to be small. Inhomogeneities are indeed captured by differences in patch volumes $V_{(i,j,k)+b} - V_{(i,j,k)-b}$, where b is a unit vector in the b-direction (in some coordinate system). The dynamics will include, in

general, interaction terms given by (complicated) polynomials of such differences. We can then quantize each patch as in (loop) quantum cosmology, e.g. in the volume representation, obtaining for each patch a wavefunction $\psi_{V_{i,j,k}}$.

The total state of the inhomogeneous universe could be then considered, treating the universe as a quantum many-body system with each patch playing the role of a fundamental atomic constituent. Notice the resonance with the TGFT framework. If inhomogeneities are small, thus correlations among patches are small, then a reasonable approximation for the total state of the universe is the product state:

$$\Psi\left(V_1, V_2, ...\right) = \frac{1}{N!}\psi_1(V_1)\psi_2(V_2)\cdots \tag{11.34}$$

where we have assumed a relabelling invariance of the patches (i.s. a bosonic statistics). This can be seen as a counterpart of diffeomorphism invariance in a discrete setting, and it is a common element in canonical loop quantum gravity and TGFTs, as well as in simplicial quantum gravity. Incidentally, this form of the total wavefunction for the universe has been suggested also in Kleinschmidt and Nicolai (2009) in the context of a canonical quantization of (symmetry-reduced) supergravity based on symmetries, for the regime close to the cosmological singularity, inspired by the BKL conjecture. In fact, if inhomogeneities are -very- small, then one could expect the quantum state of the universe to be close to an even simpler state, of the form:

$$\Psi\left(V_1, V_2, ...\right) = \frac{1}{N!}\psi(V_1)\psi(V_2)\cdots \qquad , \tag{11.35}$$

that is, a condensate state in which all atomic constituents, i.e., all homogeneous patches, have the same individual wavefunction. Notice that this is exactly the kind of

quantum geometry states used to extract the mean field approximation in TGFT cosmology, that is the coherent states of the TGFT field, now restricted to the case of a fixed 'particle number,' i.e., a fixed number of homogeneous patches (there, in TGFT, QG atoms). Following these assumptions, we now need to evaluate the quantum dynamics encoded in the Hamiltonian constraint (11.33). We can proceed as in the case of BECs. The simplest approximation is a quantum dynamics obtained from minimizing th expectation value of the Hamiltonian constraint evaluated in the full quantum state (11.35). This is akin to minimizing the energy functional in BECs. In such quantum state, the single-patch wavefunction ψ determines the whole state as a collective variable, and the resulting dynamics will necessarily be some equation for it.

Proceeding in this way, the expectation value of the quadratic term in the Hamiltonian constraint gives a quadratic functional in ψ with the homogeneous WdW operator (in loop quantum cosmology formulation) as its kernel. The interaction (higher-order) terms give instead a higher-order functional in ψ. For example, considering the interactions between pairs of homogeneous patches only:

$$W_{int}(V_1, V_2) = \alpha(V_1 - V_2)^2/V^2 \qquad \longrightarrow$$

$$\longrightarrow \qquad \langle \hat{W}_{int} \rangle_\Psi = \frac{\alpha}{V^2} \int dV_1 dV_2 |\psi(V_1)|^2 |\psi(V_2)|^2 (V_1 - V_2)^2$$

It is clear, then, that the end result of this construction is going to be a non-linear (and non-local) equation for the collective variable given by the cosmological wavefunction ψ defined on minisuperspace, with specific details depending on the initial Hamiltonian constraint operator and the various approximations ad choices made along the way. In fact, the resulting dynamics will be very close to the one presented in the previous section, coming from 3rd quantized loop quantum cosmology. In the end, we obtain

again, from a different perspective, an effective framework given by hydrodynamics on (mini)superspace, further supporting our proposal.

11.5 A Possible Upshot: A New Route for Analogue Gravity?

Before concluding, let us point out a further possible upshot of our proposal: a novel research direction for analogue gravity simulations in condensed matter systems. Specifically, it suggests that such simulations do not need to be confined to the kinematical sector of gravitational theories: the gravitational dynamics can be reproduced too, possibly beyond the classical regime.

Analogue gravity models in condensed matter systems are by now an established research area, backed up by a large body of literature.[9] They are the subject of a variety of theoretical developments, exploring the details of the gravitational interpretation of condensed matter phenomena, and of promising experimental efforts, aiming at reproducing in the lab the phenomena corresponding to often extreme (semi-classical) gravitational effects which are difficult or even impossible to test directly by astrophysical or cosmological observations. They are based on the fact that an effective curved geometry emerges naturally and generically at the hydrodynamic level in condensed matter systems, specifically classical and quantum fluids (in fact, this fact extends also to a number of non-fluid systems). More precisely, excitations over stable ground states of these systems couple to this emergent geometry, and not to the (background) laboratory metric, and their effective description is given by (relativistic) QFTs on curved spacetimes, with the speed of sound playing the role of the speed of light in the standard spacetime context. Moreover, the correspondence

is rather general also from a gravitational perspective, in the sense that the class of emergent curved geometries is broad enough to include a large number of interesting gravitational phenomena, in particular for semi-classical effects around black holes and in cosmological spacetimes of direct physical interest.

As mentioned, one reason for the attention devoted to this research area is that analogue gravitational scenarios, including exotic ones like Hawking radiation or cosmological particle production, can be reproduced (and measured) in concrete lab situations. A more theoretical reason of interest is that condensed matter systems allow investigate frontier scenarios (e.g. violation of Lorentz symmetry or other possible quantum gravity phenomenology), as well as more formal aspects of classical and quantum gravity (e.g. the role of the equivalence principle, quantum unitarity etc., of interest for fundamental physics), with (in principle) total control over both 'effective curved spacetime QFT' and the 'fundamental' atomic physics originating it. This is lacking in fundamental physics, where the underlying quantum gravity theory is not under control nor established.

However promising all of the above is,[10] one big issue limits the actual impact of results and prospects in the field, as currently understood. The gravitational dynamics is not reproduced in analogue gravity system. In fact, the (almost complete) consensus is that the gravitational dynamics just *cannot* be reproduced, at least as described by a generally covariant theory like GR. The physical reasons for this consensus are strong. They derive from the structural core of the gravitational analogy in condensed matter: the dynamics of emergent geometry is governed by the non-relativistic hydrodynamic eqns of the system. These are structurally different from GR-like eqns, and the structural differences encode crucial physical differences.

In particular: hydrodynamics is background dependent, i.e., it depends on a fixed background spacetime metric, and thus not fully dynamical from the spacetime point of view; it identifies preferred frames (isometries of background metric); it has a special global symmetry, i.e., the Schrödinger symmetry (which can be extended to the relativistic case to include Lorentz boost, rather than Galilean ones) rather than the diffeomorphism symmetry of GR and other generally covariant gravitational theories.

All these differences may be obscured or negligible at a semi-classical, perturbative level (even gravitational perturbations can in principle be reproduced in analogue systems), but eventually show up both at the theoretical level and, one should expect, the experimental one. For example, they prevent a full account of the physics of the backreaction of quantum fields on spacetime itself. This is a strong constraint on the potential development of analogue gravity simulations and of their theoretical analysis. It is also, for many, a strong reason to warrant only limited value to the many results obtained in this area, or to maintain a certain mistrust on their fundamental significance.

This is not the place to attempt any detailed analysis, and at the moment we do not have enough results to properly undermine this consensus. However, it should be clear that, at the very least, our results from Quantum Gravity (TGFT) and mathematical physics, and the relational perspective on gravitational physics, suggest otherwise, and our proposal indicates a possible way forward. Consider the derivation of cosmological dynamics, including cosmological perturbations, from quantum gravity hydrodynamics; and consider the existing maps between cosmology and BEC hydrodynamics and their matching symmetries. These and other results in the literature show that hydrodynamic equations can, in fact, be mapped to relativistic cosmological dynamics, which is

something argued to be impossible in the analogue gravity context. Similarly, the two dynamical contexts have in fact the same symmetries, once properly compared.

The key point is to accept a shift in perspective. The hydrodynamics should be understood as defined on (mini)superspace (the space of field values), and not on the 'spacetime manifold,' and the relevant symmetries are field transformations on (mini)superspace, and not on the 'spacetime manifold.' Then the reliance on a metric background with special isometries is not in contradiction with the general covariance and background independence (from the point of view of spacetime physics) of GR: (mini)superspace is a metric manifold with a fixed metric, the supermetric, which has non trivial isometries, those corresponding to the field symmetries of the gravitational dynamics. Thus, it may be possible to have a proper gravitational dynamics (for both background universe and perturbations) in quantum many-body systems, in a hydrodynamic approximation of the same, after all, provided one changes to a relational or "superspace-based" perspective on spacetime physics, rather than assigning a central role to the 'spacetime manifold,' which is, in fact, a redundant mathematical structure.

The suggestion for analogue gravity simulations would be, then, to try to reproduce in the lab not the 'spacetime manifold,' its metric and the fields living on top of it, but (mini)superspace with its (super) metric, and the appropriate hydrodynamic system living on it. Having done so, cosmological observables and, more generally, gravitational dynamics should be reconstructed in a second step, from appropriate hydrodynamic averages, rather than identified directly with hydrodynamic perturbations. Specifically, one could focus on the examples of minisuperspace hydrodynamics discussed in the previous sections, that have been already shown to allow a reconstruction of gravitational dynamics, and on any other

such models as suggested, e.g., by quantum gravity considerations. We leave a more detailed elaboration of the above points to another contribution, and the proper implementation of the suggestion in an analogue gravity context to future work.

11.6 Conclusions

We have put forward a proposal for an effective framework to study cosmological physics, incorporating quantum gravity features of our universe, given by hydrodynamics on minisuperspace (the space of field configurations ‘at a point’ or the space of homogeneous fields). This hydrodynamics on minisuperspace amounts to a non-linear and non-local extension of quantum cosmology, but with a very different perspective. It is understood, in fact, as the result of some coarse graining of quantum gravity degrees of freedom, which may well be non-spatiotemporal and non-geometric, rather than of a straightforward quantization of a symmetry-reduced sector of a classical gravitational theory (thus, while there may be an underlying Hilbert space of ‘QG atoms,’ there is no Hilbert space of homogeneous universes, as in quantum cosmology).

We have outlined the conceptual basis of the proposal, built on a relational perspective on spacetime and on the idea of spacetime emergence, combined with a vision of the universe as a peculiar quantum many-body system. We then offered some evidence in support of the proposal. We started from a number of results in mathematical physics, unravelling a surprising correspondence between cosmological dynamics and standard hydrodynamics. Then, we summarized recent results in one quantum gravity formalism, i.e., tensorial group field theory (TGFT), showing how cosmological dynamics emerges from its hydrodynamics sector, defined on minisuperspace. The tensorial group field theory formalism is itself a

convergence of several quantum gravity formalisms, suggesting that the role of hydrodynamics on minisuperspace as the relevant effective framework could be more general. In further support of this hypothesis, we have outlined two other ways in which the same framework can be derived, in a quantum gravity context, one following the idea of 3rd quantization of gravity, as an effective way of accounting for topology change, and another based on techniques from theoretical cosmology as well as (loop) quantum cosmology, as an effective way to deal with cosmological inhomogenities. While the research programme of TGFT cosmology is rather established and already progressing fast, with more results yet to come but already within the radar, much more work is needed to establish the proposed framework as a universal one across quantum gravity approaches. Such universality, if shown convincingly, will add to the theoretical interest in the proposed framework, as well as to the robustness of its predictions of new cosmological physics. Beyond the theoretical developments, it is the direction of physical cosmology, in fact, that should be pursued intensively, in order to prove the usefulness of the proposed framework in producing testable predictions of new cosmological phenomena, of possible quantum gravity imprints in them, or novel explanations of existing cosmological puzzles. The implementation of the suggested new route for analogue gravity simulations reproducing also the gravitational dynamics should be, of course, another objective to pursue. Along any of the above research directions, we expect an exciting dialogue between quantum gravity, the theory of quantum fluids and cosmology.

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Footnotes

1 Minisuperspace is mathematically under control. For a general discussion of the much more complicated ‘superspace,’ i.e., the full configuration space of (canonical) GR, see Giulini (2009).

2 We restrict consideration to the 4-dimensional case.

3 One could extend the dynamical variables to include lapse and shift functions, in a canonical decomposition of the 4-metric. However, these are not dynamical degrees of freedom.

4 Authors, in preparation.

5 Authors, in preparation.

6 Authors, in preparation.

7 Authors, in preparation.

8 Indeed, let us mention that effective continuum equations of this type, and in fact a formalism very similar to a simplified version of TGFTs, have been obtained also in the context of 2d simplicial quantum gravity formulated via causal dynamical triangulations, as the effective continuum limit of the lattice path integral extended to a sum over topologies, in Ambjørn et al. (2009).

9 See Barceló et al. (2005) for a review.

10 See again Barceló et al. (2005) for a proper discussion of achievements, current research and open issues.

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12. Arrows of Time in Bouncing Cosmologies

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Abstract
Different approaches to quantum gravity, such as loop quantum cosmology and group field theory, predict the resolution of the initial cosmological singularity via a ‘bounce’: a regular spacetime region that connects the expanding branch of the universe to a contracting branch. The cosmological arrow of time, which by definition points in the direction of cosmic expansion, is reversed at the bounce. Nonetheless, it is still possible to discriminate between the two branches by considering different arrows, as defined for instance by the growth of perturbations. After reviewing general aspects of the time arrow problem in cosmology, we examine the properties of different arrows of time in bouncing cosmologies, focusing on the loop quantum cosmology bounce as a case study. These issues are examined in detail for an exact solution to the

effective Friedmann equations of loop quantum cosmology with pressureless dust and a cosmological constant, which is a simplified version of the Λ CDM bounce scenario.

12.1 Arrows of Time and Their Cosmological Origin

The physical world offers a wealth of examples of phenomena taking place in a definite temporal direction, which are either very rarely or never spontaneously reversed. Despite the reversibility of fundamental physical laws at a microscopic level,[1] the evolution of macroscopic physical systems is typically characterised by an unequivocal directionality of time that allows us to discriminate the past from the future. This basic fact permeates virtually every aspect of our experience. For instance, we never observe eggs uncracking, footsteps appear in the sand after the retreating tide, or black holes disappear into uncollapsed matter. Several distinct *arrows of time* can be identified in nature, based on intrinsic temporal asymmetries observed in different classes of physical phenomena (Hawking 1985; Gell-Mann & Hartle, 1991):

- The thermodynamical arrow of time, pointing in the direction of increasing entropy;
- The psychological arrow of time, according to which we can only remember our past but not our future;
- The electrodynamical arrow of time, explaining why only retarded solutions of the Maxwell equations are observed but never advanced ones;
- The cosmological arrow of time, pointing in the direction of cosmic expansion. In standard cosmology based on general relativity, the arrow always points away from the initial Big Bang singularity if the universe is flat or open,

whereas in a closed universe the arrow may reverse its direction at late times if the cosmological constant is smaller than a certain critical value. The situation is different in bouncing cosmologies, where the cosmological arrow is reversed at the bounce and points away from it on either side;

- The arrow of time specified by the direction in which density perturbations grow, which ultimately determines the formation of structures in the universe.

Further arrows of time are also defined by, e.g., state vector reduction in the measurement of a quantum system, diffusive and radiative processes in astrophysics, biological processes, and so on (Ellis 2013). Although conceptually distinct, the different arrows of time are not independent of one another. For instance, the psychological arrow clearly depends on the thermodynamical one, as biological systems obey the laws of thermodynamics. The electrodynamical arrow is determined by a boundary condition of no incoming radiation in the past, which in the Wheeler–Feynman absorber theory is traced back to the thermodynamic properties of the absorbing medium (Hawking, 1985; Zeh, 2011; Forgione, 2022). More in general, if the physical descriptions of systems are organized hierarchically, ranging from the smallest to the largest scales through successive levels of emergence, it is possible to identify relations between different levels that proceed in both a bottom-up and a top-down manner (Ellis, 2013). We stress that, even for physical systems which are governed by time-symmetric equations, in general we do not expect that solutions enjoy the same symmetry as the equations. Rather, solutions would come in pairs that are the image of one another under time reversal. The reason why only certain such solutions are physically realized, but not their time-reversed counterpart, ultimately hinges on the boundary conditions imposed on the system at hand,

which are typically determined by the environment (Zeh, 2011).

The arrows of time corresponding to different levels of description of the physical reality are locally determined, and thus may in principle point in opposite directions in different spacetime regions.[2] However, each of the arrows happen to consistently point in the same direction across the observable universe: this constitutes the so-called *locality issue* (Ellis, 2013). The remarkable agreement between the different arrows of time as well as their spatiotemporal coherence over vastly different scales suggest that they may have a common origin. Thus, one may speculate that the cosmological arrow of time in the early universe played the role of a *global master arrow*, from which all other arrows can be ultimately derived (Ellis, 2013; Kiefer, 2011).

According to an early speculative proposal by Gold (1962), the thermodynamical arrow of time may coincide with the cosmological one *at all times*. In a closed universe, this would imply the reversal of the thermodynamical arrow at the turnaround point, where the universe enters the recollapse phase. All physical processes would then run in the opposite time direction, and all memories and records created during the expanding phase would be erased. A realization of this scenario necessarily requires low entropy boundary conditions both at the Big Bang and Big Crunch. However, this proposal has not been generally accepted and faced severe criticism (Penrose, 1979; Zeldovich & Novikov, 1983). It was argued by Hawking (1985) that a scenario similar to the Gold universe could be naturally realized in the no-boundary proposal for the path integral in quantum cosmology, although Page (1985) later showed that this is not necessarily the case. In the Wheeler–DeWitt approach to quantum cosmology, a natural choice of ‘time’ variable is represented by (the logarithm of) the cosmic

scale factor; hence, it is natural to impose boundary conditions in the limit where the universe is small (i.e., the 'far past' for such a choice of time variable), which corresponds to both initial and final cosmological singularities in a closed universe. Imposing low-entropy boundary conditions in the limit where the scale factor is small then leads to a thermodynamical time arrow that agrees with the cosmological one and reverses at the point of recollapse (Kiefer & Zeh, 1995).

It has been pointed out by Penrose (1979, 1989, 2006) that the entropy of the observable universe is much smaller by several orders of magnitude compared to its maximal value. The latter can be estimated as the Bekenstein-Hawking entropy of a spherical black hole with the same total mass as the observable universe. In a universe with baryon number 10^{80}, the maximal entropy is about $S_{\max} \approx 10^{123}$ in natural units,[3] whereas the entropy of the cosmic microwave background (CMB) radiation is $S \approx 10^{88}$. The extremely small value of S relative to $S_{\max}$ implies that the universe must have started in a rather special state at the Big Bang. Moreover, CMB observations show that, while matter was in a thermal (i.e., maximal entropy) state at the time of recombination, the gravitational field was extremely uniform and thus in a low entropy state.[4]

This motivated the conjecture (*Weyl curvature hypothesis*) that in our universe the Weyl curvature vanishes in the approach to initial spacetime singularities, and therefore gravitational degrees of freedom are not excited at the Big Bang (Penrose, 1979, 1989, 2006). On the other hand, final spacetime singularities are unconstrained. This hypothesis ensures the smoothness of the initial state of the universe, thus providing a low-entropy past boundary condition that explains the existence of a thermodynamical time arrow pointing in the direction of cosmic expansion (Penrose, 1994). We note that, in

general, solutions to the Einstein equations would display chaotic Belinskii-Khalatnikov-Lifshitz (BKL) oscillatory behaviour in the approach to the initial cosmological singularity, which implies a highly anisotropic and inhomogeneous initial state (Belinsky et al., 1982). Thus, the Weyl curvature hypothesis constitutes indeed a strong constraint on the initial state.

The Weyl curvature hypothesis represents a particular instance of the so-called *Past Hypothesis* (Albert, 2000): in order to explain the smallness of the entropy of the universe and its increase with cosmic expansion, it seems necessary to assume a special initial state with an extremely low entropy. We note that in a universe where the second law of thermodynamics holds, it is impossible to retrodict the initial state. In fact, as a consequence of the time-reversal invariance of the microscopic dynamics, standard arguments from statistical mechanics show that the evolution from the present macrostate will lead to entropy increase *both* into the future and into the past (Penrose, 1994; Schiffrin & Wald, 2012). Since we know from cosmological observations that the entropy of the universe actually decreases towards the past, retrodictions are bound to give incorrect answers. It is therefore necessary to add further theoretical input on the initial state and use the latter to predict the future, rather than attempt to retrodict the past. The status of the Past Hypothesis as a possible solution of the low-entropy puzzle in cosmology has been much debated in the literature; however, some difficulties have been pointed out concerning both its precise formulation and the explanatory power of statistical arguments.[5] used in the proposal (Price, 2004; Callender, 2004; Earman, 2006; Gryb, 2021).

Alternatively, attempts have been made to explain the low entropy of the observable universe starting from a

random initial state as the result of a dynamical process, such as inflation. However, special initial conditions must be assumed at the onset of inflation in order to produce an inflating patch that evolves to a universe like ours; thus, the issue of the exceptionality of the initial state is just shifted to the pre-inflationary epoch (Wald, 2006; Earman, 2006; Davies, 2014). Yet another approach to explain the initial conditions of the universe is to appeal to quantum cosmology (Vilenkin, 1983; Hartle & Hawking, 1983; Feldbrugge et al., 2017), where the problem translates into the identification of appropriate boundary conditions in superspace, or, equivalently, of specific integration contours in the path integral over geometries. There are also proposals where entropy grows symmetrically both to the past and to the future of a given low-entropy spatial hypersurface (Aguirre & Gratton, 2003; Carroll & Chen, 2004; Vilenkin, 2013).[6] Two-times *high-entropy* boundary conditions can also lead to intermediate low entropy states, as shown in Deutsch and Aguirre (2022). Going beyond standard cosmology, finding an explanation for the low entropy of the universe remains an open problem also in alternative early universe scenarios. In the remainder of this contribution we will focus specifically on the issue of determining arrows of time in bouncing cosmologies, taking loop quantum cosmology as a case study.

12.2 The LQC Bounce

It is well-known that classical general relativity predicts its own breakdown at spacetime singularities (Hawking & Penrose, 1970). There is a widespread expectation that such singularities must be resolved in a theory of quantum gravity. In the cosmological case, this idea has been realized concretely in the framework of non-perturbative approaches to quantum gravity, such as loop quantum

cosmology (LQC) (Ashtekar & Singh, 2011) and group field theory (GFT) (Oriti, 2006; Oriti et al., 2016), where the initial cosmological "Big Bang" singularity is replaced by a "Big Bounce" (see also Oriti's contribution in this volume and the reviews Pithis & Sakellariadou, 2019; Gabbanelli & De Bianchi, 2021).[7]

In these models, a nonsingular bounce is obtained as a result of quantum gravity effects, which cause the gravitational interaction to turn into a repulsive force at high energy densities, typically of the order of the Planck density. The bounce represents a smooth spacetime region that connects the expanding branch of the universe to a past contracting branch. A similar scenario has been realized also in Pre-Big Bang cosmology in the framework of string theory (Gasperini & Veneziano, 2003, 2023; Veneziano, 1999), see also (Conzinu et al., 2023). Besides quantum gravity, singularity resolution has also been studied in the framework of modified gravity, where early proposals were based on the implementation of the so-called limiting curvature hypothesis (Markov, 1982; Mukhanov & Brandenberger, 1992), and bouncing cosmologies have been concretely realized more recently in different classes of theories (including, e.g., scalar-tensor theories, massive gravity, and higher-order modifications of general relativity, see (Easson et al., 2011; Cai et al. 2012a, b; Ijjas & Steinhardt, 2016; Chamseddine & Mukhanov, 2017; de Cesare 2019a, b; de Cesare & Wilson-Ewing, 2019; Ilyas et al., 2020 and references therein). Bouncing models resolve the classical horizon problem of standard cosmology, and may represent either an alternative to the inflationary mechanism for the generation of primordial cosmological fluctuations or a complement to inflation that includes the effects of Planck scale physics (Brandenberger & Peter, 2017; Agullo et al., 2023).

In this section we focus specifically on the LQC bounce. After reviewing the effective cosmological dynamics, we will consider a concrete model where the energy content consists of pressureless dust and a cosmological constant, and derive an exact solution of the LQC effective Friedmann equations. We will show that no physically meaningful time asymmetry exists at the background level, i.e., both the cosmological and the thermodynamical arrows of time are collinear and point away from the bounce. The dynamics of perturbations and the resulting arrow of time will be studied in the following section. The effective dynamics for a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) model are (Ashtekar & Singh, 2011):

$$H^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right) \ , \tag{12.1a}$$

$$\dot{H} = -4\pi G(\rho + p)\left(1 - 2\frac{\rho}{\rho_c}\right) \ . \tag{12.1b}$$

Here $H = \dot{a}/a$ is the Hubble expansion rate, a is the scale factor, ρ and p are the energy density and pressure of matter, respectively. Evolution is defined with respect to proper time t and an overdot denotes time derivative $\dot{} = \mathrm{d}/\mathrm{d}t$. The critical energy density is Planckian, $\rho_c \sim \rho_{\mathrm{Pl}}$. From Eqs. (12.1a), (12.1b) follows the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{12.2}$$

We assume that $\rho \geq 0$ and that the null energy condition is satisfied $\rho + p \geq 0$. Equation (12.2) then implies that ρ is monotonically decreasing when regarded as a function of a. This implies in particular that the maximum of ρ is attained when a is minimized. The bounce takes place in the early universe when the energy density of matter is at a maximum $\rho = \rho_c$, whereby the Hubble rate vanishes $H = 0$

and the scale factor attains its minimum. The initial cosmological singularity is fully resolved.[8] Moreover, Eq. (12.1b) implies that $\dot{H} > 0$ in a neighbourhood of the bounce where $\rho_c/2 < \rho \leq \rho_c$, whereas $\dot{H} < 0$ if $\rho < \rho_c/2$. Standard cosmology is recovered in the regime where $\rho \ll \rho_c$, which is realized both in the late-time and early-time limits.

Note that it is also possible to interpret Eqs. (12.1a), (12.1b) as the Friedmann equations for an effective fluid with $\tilde{\rho} = \rho(1 - \rho/\rho_c)$ and $\tilde{p} = p(1 - 2\rho/\rho_c) - \rho^2/\rho_c$. The effective fluid violates the null energy condition in a neighbourhood of the bounce, where $\tilde{\rho} + \tilde{p} < 0$ and as a consequence $\dot{H} > 0$. This allows for the change of sign of the Hubble rate, i.e., the transition from contraction to expansion. The bounce (and thus the ensuing violation of the null energy condition for the effective fluid) has a very short duration, of the order of the Planck time, which is typical in models motivated from quantum gravity.

Without loss of generality, we can shift the origin of time so that the bounce takes place at $t = 0$. With this choice, the universe is in a contracting phase for $t < 0$ (where $H < 0$), whereas for $t > 0$ it is expanding ($H > 0$). Also note that at this level the distinction between 'past' and 'future' is merely a matter of conventions, and can be reversed by changing the orientation of the time axis. In this and the following section we will investigate whether any *physical* time asymmetries emerge, which may enable us to differentiate between the past and the future.

For the sake of concreteness, we consider a model with a cosmological constant $\Lambda > 0$ and a pressureless fluid $\rho = \rho_{\rm m} + \Lambda/(8\pi G)$, with $\rho_{\rm m} = \rho_o(a_o/a)^3$, where a_o is the scale factor at the bounce. The presence of a positive cosmological constant ensures that the universe is asymptotically de Sitter, and therefore one can define cosmological event horizons (as well as their entropy). The

present model is a simplified version of the ΛCDM bounce scenario in LQC, which includes also the contribution of radiation (which dominates over dust in the early universe) and has been studied earlier in Cai and Wilson-Ewing (2015) and Cai et al. (2016). In turn, the ΛCDM bounce is a generalization of the LQC matter bounce scenario (Wilson-Ewing, 2013; for a recent review see Agullo et al., 2023).

Using Eq. (12.1a) at the bounce, we obtain $\rho_o + \Lambda/(8\pi G) = \rho_c$, which can be used to eliminate ρ_o. The model is exactly solvable,[9] with scale factor given by

$$a(t) = a_o \left[1 + \frac{4\pi G\rho_c}{\Lambda}\Big(\cosh\left(3H_\infty t\right) - 1\Big)\right]^{1/3}, \tag{12.3}$$

where we defined $H_\infty \equiv \sqrt{\Lambda\left(1 - \Lambda/(8\pi G\rho_c)\right)/3}$. This solution of the effective Friedmann equations in LQC has been obtained earlier in (Chinaglia et al., 2017). In our universe the cosmological constant is much smaller than the Planck curvature scale $8\pi G\rho_c \sim \ell_{\rm Pl}^{-2} \gg \Lambda$. Therefore, we have approximately $H_\infty \approx \sqrt{\Lambda/3}$ for the value of the Hubble rate at late times. From the exact solution (12.3) we can obtain the Hubble rate as a function of time

$$H(t) = \frac{H_\infty \sinh\left(3H_\infty t\right)}{(\Lambda/4\pi G\rho_c) + \cosh\left(3H_\infty t\right) - 1}. \tag{12.4}$$

The duration of the bounce, computed as the duration of the time interval where $\dot{H} > 0$, is $T \approx \sqrt{2/(3\pi G\rho_c)} \sim t_{\rm Pl}$. Note that in the $\Lambda \to 0$ limit, the universe is filled with pressureless dust and thus we recover the LQC matter bounce solution (Wilson Ewing, 2013)

$$a(t) \approx a_o\left(1 + 6\pi G\rho_c t^2\right)^{1/3}, \quad H(t) \approx \frac{t}{(4\pi G\rho_c)^{-1} + \frac{3}{2}t^2}. \tag{12.5}$$

In all bouncing models, the cosmological arrow of time determined by the background reverses its direction at the

bounce, since by definition it always points in the direction of cosmic expansion. We now turn our attention to the thermodynamical arrow, as determined by the increase of horizon entropy. The future and past event horizons[10] are defined respectively as:

$$R_h^+(t) = a(t)\int_t^{+\infty} \frac{\mathrm{d}\ t'}{a(t')}\ , \qquad R_h^-(t) = a(t)\int_{-\infty}^{t} \frac{\mathrm{d}\ t'}{a(t')}. \tag{12.6}$$

The presence of a positive cosmological constant in the model at hand ensures that the integrals in Eq. (12.6) are convergent. From the definitions (12.6), we obtain that the horizons obey the following differential equations

$$\dot{R}_h^+(t) = H(t)R_h^+(t) - 1\ , \quad \dot{R}_h^-(t) = H(t)R_h^-(t) + 1. \tag{12.7}$$

In the regime where $\dot{H} > 0$, both horizons are increasing in the expanding phase $H > 0$ and decreasing during contraction $H < 0$; moreover, they both lie outside the Hubble radius, $\dot{R}_h^{\pm}(t) \geq 1/|H(t)|$. Asymptotically, a de Sitter universe is approached both in the distant past and future, and one has $\lim_{t\to+\infty} R_h^+ = \lim_{t\to-\infty} R_h^- = H_\infty^{-1}$. The behaviour of the horizons in the $\dot{H} > 0$ regime is consistent with the general analysis in Davies (1988). However, in the proximity of the bounce, where the null energy condition is violated by the effective fluid $\tilde{\rho} + \tilde{p} < 0$, such monotonicity properties of the horizons no longer hold. This implies in particular that the minimum value of the horizons is not attained at the bounce at $t = 0$.

Given the above, in the following we will limit ourselves to the $\dot{H} > 0$ regime. Here the interpretation of the future event horizon as the entropy of the cosmological background is justified (Davies, 1988); therefore, the entropy is $\mathcal{S}^+(t) \propto (R_h^+(t))^2$ and increases with $|t|$, both in the positive and the negative time directions. However, there appears to be a temporal asymmetry, since the entropy $\mathcal{S}^+(t)$ is not an even function of time in this model. Naively,

this may seem to suggest the existence of a thermodynamical arrow of time for the cosmological background. However, this conclusion is not quite correct, since the apparent time asymmetry merely originates from the definition (12.6) of the event horizon, which is not itself time-symmetric. When time is reversed, also the future and past event horizons must be interchanged. In fact, the entropy of the time-reversed solution (obtained under the transformation $t \to -t$) is given by the area of the *past* event horizon $\mathcal{S}^{-}(t) \propto (R_h^{-}(t))^2$. In complete analogy with the previous case, the entropy increases with $|t|$. Furthermore, as a consequence of the symmetry of the background solution (12.3) under time reversal, we have $R_h^{-}(t) = R_h^{+}(-t)$, which in turn implies $\mathcal{S}^{-}(t) = \mathcal{S}^{+}(-t)$. Thus, in the model at hand the entropy is indeed invariant under time reversal. Therefore, similarly to the cosmological arrow of time discussed earlier, we conclude that also the thermodynamical arrow of time points in opposite directions in the contracting and expanding branches. If we restrict our attention to the homogeneous and isotropic background (perhaps unsurprisingly) there are no time asymmetries that may enable us to distinguish the two branches. In the next section, we will discuss how time asymmetries do in fact emerge in the evolution of cosmological perturbations.

12.3 Time Arrow of Perturbations in the ΛCDM Bounce Scenario

We consider the ΛCDM bounce scenario of Cai and Wilson-Ewing (2015) and Cai et al. (2016) and review the evolution of scalar perturbations in their model. As in the simplified model considered in the previous section, also the ΛCDM bounce is symmetric. The difference between the two models lies in the fact that the ΛCDM bounce also takes

into account the contribution of radiation, which dominates the energy density of the universe at the bounce. Moreover, there is an intermediate era of radiation domination between the bounce and the matter dominated era. At late times, as well as in the distant past, radiation becomes sub-leading compared to non-relativistic matter and dark energy, and therefore the evolution of the scale factor in this regime is well approximated by (12.3). In both models, general relativity is recovered in the large $|t|$ limit.

The evolution of scalar primordial fluctuations in Fourier space is governed by the Mukhanov–Sasaki equation (Cai & Wilson-Ewing, 2015)

$$v''(\eta) + \left[c_s^2 k^2 \left(1 - \frac{2\rho}{\rho_c}\right) - \frac{z''(\eta)}{z(\eta)}\right] v(\eta) = 0 \tag{12.8}$$

where k is the comoving wave number, $z = a\sqrt{\rho + p}/(c_s H)$ (with c_s denoting the speed of sound), and primes denote derivatives with respect to conformal time $\eta = \int \mathrm{d}\, t/a(t)$. The integration constant in the definition of η can be chosen in such a way that $\eta = 0$ at the bounce. The Mukhanov–Sasaki is related to the comoving curvature perturbation as $v = z\mathcal{R}$.

The equation of motion (12.8) is manifestly invariant under time reversal $\eta \to -\eta$. In fact, it follows from the definition that z is odd under time reversal, which in turn implies that the potential z''/z is an even function of η. Therefore, asymmetric solutions for $v(\eta)$ may only originate from initial conditions that explicitly break time-reversal symmetry. In fact, in order to get phenomenologically viable predictions, initial 'vacuum' boundary conditions are imposed on the perturbations in the distant past $\eta \to -\infty$, where one assumes the asymptotics $v(\eta) \approx e^{-ic_s k\eta}/\sqrt{2c_s k}$. Modes that are observationally relevant become super-horizon during matter domination in the contracting phase, stay in this regime during radiation domination and

throughout the bounce, and eventually re-enter the horizon during the expanding phase. To obtain an almost scale-invariant spectrum, it is important to ensure that observable modes exit the horizon during the matter dominated contraction. A slight red tilt originates from a small negative contribution to the pressure due to the non-zero cosmological constant. Furthermore, the effective equation of state of the fluid with $\rho = \rho_{\rm m} + \Lambda/(8\pi G)$, $p = -\Lambda/(8\pi G)$ is approximately constant during the ΛCDM-dominated epoch when observationally relevant modes exit the horizon, i.e., one has $p = w_{\rm eff}\rho$ with constant $w_{\rm eff} < 0$ to a good approximation (Cai & Wilson-Ewing, 2015).

The evolution of perturbations in this model has been studied using both analytical and numerical methods in Cai and Wilson-Ewing (2015), to which we refer the reader for more details. After the bounce and at horizon re-entry, the leading contribution to the curvature perturbation is a constant mode. The scalar power spectrum reads

$$\Delta^2_{\mathcal{R}} = \frac{k^3}{2\pi^2}|\mathcal{R}|^2 \propto \sqrt{\frac{\rho_c}{\rho_{\rm Pl}}}\frac{H_e\ell_{\rm Pl}}{c_s^3}\left(\frac{8\pi G\rho_c|H_e|}{3k^4}\right)^{-3w_{\rm eff}}, \qquad (12.9)$$

where H_e is the Hubble rate at the time of matter-radiation equality in the contracting branch. The temporal asymmetry in the evolution of perturbations in the model at hand is clear from the numerical simulations (Cai & Wilson-Ewing, 2015). Moreover, although perturbative modes that were originally in the sub-horizon regime in the distant past will eventually re-enter the horizon in the expanding branch, some of them (i.e., those that are relevant for structure formation) will then grow large in amplitude during matter domination, so that linear cosmological perturbation theory is no longer applicable and the evolution becomes nonlinear. This indicates that the temporal asymmetries that already emerged at the linear level due to an asymmetric choice of boundary conditions

will eventually lead to a strong asymmetry between the contracting and expanding branches due to the growth of nonlinearities. Therefore, perturbations can determine an arrow of time even in scenarios with a symmetric bounce. Furthermore, perturbations that are both sub-Hubble and super-Jeans during the matter-dominated era in the contracting branch may become non-linear and, under certain conditions on the speed of sound, lead to the formation of black holes that could propagate through the bounce to the expanding branch (Quintin & Brandenberger, 2016). The increase in entropy due to black-hole formation determines a thermodynamical arrow of time which agrees with the one defined by the growth of inhomogeneities.

12.4 Discussion

We have seen that a number of different arrows of time can be introduced in bouncing cosmologies. At the background level, the cosmological arrow of time always points in the direction of expansion, and therefore away from the bounce. It is also possible to define a thermodynamical arrow of time, pointing in the direction of horizon entropy growth. However, if the bounce is symmetric also the background thermodynamical arrow is unable to introduce any distinction between the two branches. We examined this issue in detail for an exact solution of the effective Friedmann equations of LQC, describing a bouncing cosmology with pressureless dust and a positive cosmological constant. This simple example shows that, in general, the analysis of the evolution of the cosmological background is not sufficient to draw any reliable conclusions about the arrow of time. We have then examined the evolution of scalar perturbations and discussed how the time arrow of inhomogeneities growth originates from initial conditions that break time-reversal

symmetry. A specific choice of (vacuum) initial conditions is made in the distant past, which amounts to a Past Hypothesis. This plays an analogous role to Penrose's Weyl hypothesis in standard cosmology in selecting a special initial state for the universe.

Further arrows of time can be introduced in the case of asymmetric bounces, which we have not discussed in this contribution. For instance, the bounce may be preceded by an epoch of ultra-slow contraction, as in ekpyrotic models, or there could be an epoch of inflation following the bounce. In both cases, such early universe scenarios single out a preferred time direction that discriminates the past from the future. This is best understood by introducing some deviations from the perfect flatness and isotropy of the FLRW cosmological background. Both in the cases of ekpyrotic and inflationary models there is only one direction of time evolution that leads to the suppression of shear anisotropies and spatial curvature starting from generic initial conditions for the homogeneous background, as ensured by the cosmic no-hair theorems (Barrow, 1987; Erickson et al., 2004). More precisely, the arrow of time points in the direction of ekpyrotic contraction and inflationary expansion—a reversed time arrow would be incompatible with the observed properties of our universe.

We stress that the issue of determining an arrow of time is common to all early universe scenarios—not just bouncing cosmologies. In this contribution, the dynamics of the cosmological background and perturbations in bouncing models has been studied at a classical effective level, focusing on a generalization of the matter bounce scenario in LQC. We have seen that in this scenario the arrow of time follows from a specific choice of initial conditions for cosmological perturbations, which are imposed in the distant past in the contracting branch and serve as an example of the Past Hypothesis. However, in order to fully address the arrow of time problem one should

work in a full theory of quantum gravity, for which we lack a complete formulation at present. To complicate matters further, in quantum gravity the time arrow problem is also closely connected with the *problem of time*, i.e., the absence of time at a fundamental level (Kuchař, 2011).

Lastly, if spacetime emerges dynamically from a pre-geometric structure, as argued for instance within the context of GFT and in the *causal sets* approach to quantum gravity (see contributions by Oriti and Forgione in this volume, and De Bianchi & Gabbanelli, 2023), the problem of defining an 'initial state' for the universe may require a reframing that makes no reference to the emergent spacetime geometry. Within this framework, it would be interesting to investigate whether the initial conditions of the universe could be explained from first principles, for instance by means of a dynamical mechanism, or if a Past Hypothesis (i.e., a law-like prescription similar to the Weyl curvature hypothesis) is still required.

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Footnotes

1 More precisely, local relativistic quantum field theories are invariant under CPT, i.e., the combination of time reversal (T), charge conjugation (C), and parity (P). This ensures the absence of a microscopic arrow of time.

2 This possibility, though speculative, is logically consistent and has been explored in Schulmann (1999) in a toy model with two weakly-coupled systems with opposite-pointing thermodynamical time arrows, realized by means of two-time boundary conditions (see also Zeh, 2005 for a critical discussion).

3 This is actually a lower bound for the maximal entropy since dark matter has not been taken into account (Penrose, 2006).

4 Although a general definition of gravitational entropy is still not available (with the exception of specific systems, such as black holes and homogeneous cosmological models, see Bardeen et al., 1973, Davies, 1988, Jacobson & Parentani, 2003), a qualitative relation between entropy increase and gravitational clumping has been pointed out by Penrose (1979, 1989). Qualitatively, the thermodynamic behaviour of a system of particles that interact gravitationally is opposite to that of a gas, due to the long range nature of the gravitational force. Spatially uniform distributions of matter correspond to low entropy states for the gravitational field. As matter particles start to clump together to form gravitationally bound structures, the entropy grows. Eventually, black holes would form and the entropy reaches its maximum value when the totality of matter collapses into a single black hole. Since the absence of clumping roughly corresponds to the vanishing of the Weyl tensor, Penrose (1979) conjectured that the latter may be a measure of entropy (see also Husain, 1988). However, this proposal faced some difficulties by Bonnor (1987) and Goode et al. (1992). See also Clifton et al. (2013).

5 One of the reasons is that the interpretation of the Liouville measure on the space of solutions for cosmological spacetimes as a probability measure faces a number of well-known difficulties (Schiffrin & Wald, 2012; Curiel, 2015). A well-defined probability measure would be needed to put on solid ground typicality arguments that are usually made when invoking the Past Hypothesis.

6 An alternative proposal with bidirectional arrows of time representing growth of complexity was made in (Barbour et al., 2014).

7 See also (Barca et al., 2021).

8 Although (harmless) weak curvature singularities may still exist (Singh, 2009).

9 A generalization of this solution can be obtained in a model with a cosmological constant and a fluid with arbitrary equation of state w (in which case the energy density is $\rho = \rho_w + \Lambda/(8\pi G)$, with $\rho_w = \rho_o(a_o/a)^{3(w+1)}$). Also in this case there exists an exact solution of the LQC effective Friedmann equation (12.1a) with a very similar structure as (12.3), given by $a(t) = a_o\left[1 + \frac{4\pi G\rho_c}{\Lambda}\left(\cosh\left(3(1+w)H_\infty t\right) - 1\right)\right]^{\frac{1}{3(1+w)}}$, see (Chinaglia et al., 2017).

10 Note that the definition of $R_h^-(t)$ coincides with the *particle horizon* used in the cosmology literature. In standard cosmology the lower integration limit corresponds to the initial cosmological singularity, which takes place at a finite value for t. However, the model at hand is non-singular and therefore the lower integration limit can be pushed to the infinite past (Davies, 1988; Faraoni, 2015).

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13. Noether Symmetries in Quantum Cosmology: A Selection Rule for Observable Universes

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Abstract
Noether's symmetries play a prominent role in any branch of physics. Here, we discuss their role in Quantum Cosmology. In particular, in this Chapter, we consider minisuperspace cosmological models, showing that the existence of symmetries, and then conserved quantities, not only allows to obtain exact cosmological solutions but also gives a selection rule through which one can recover correlations among cosmological parameters, that is a criterion to select observable universes. In this sense,

symmetries are related to physically reliable models. Examples are worked out starting from some theories of gravity.

13.1 Introduction: Symmetries and Conservation Laws

The concept of symmetry evolved and improved along the last century: the transition from global continuous symmetries (such as translation and rotation) to local continuous symmetries, and, in particular, local gauge symmetries, the addition of the internal symmetries to the space-time symmetries, the connection between charge conjugation and time reversal with the discrete symmetry of parity, symmetries in Quantum Field Theory point to the fact that the concept of symmetry is unavoidable in mathematics, physics and other sciences (Brading, 2002). The *leitmotiv* of this interest is that symmetries became a key tool for the construction of any theory at the fundamental level. For example, it is worth noticing the role of the Principle of Relativity in Special Relativity and the role of covariance as a symmetry principle in General Relativity, as well as the role of groups in Quantum Field Theory or in Particle Physics. In particular, the symmetry principle plays a fundamental role in the construction of the Standard Model of Particle Physics.

In this context, the work by Emmy Noether is of central importance since she provided a general proof that from the continuous symmetries of a given theory we can derive the conservation laws of that theory. In particular, she discovered the relation between symmetries of dynamical systems and their first integrals, i.e., physical quantities which remain constant along the evolution of the system and refer to fundamental physical quantities such as energy, linear momentum, angular momentum and so on

(Noether, 1918; Fatibene et al., 2010). Recently, Noether's symmetries assumed a prominent role also in Cosmology because they allow to select cosmological models whose parameters are observable quantities (Capozziello et al., 1996; Bajardi & Capozziello, 2022; Capozziello & Bajardi, 2022). This statement means that Noether's symmetries are capable of selecting physical models in a given class of theories (Benetti et al., 2019).

This issue is crucial in Quantum Cosmology, the branch of cosmology aimed to select initial conditions from which classical (and then observable) universes can emerge. Specifically, according to the Hartle criterion (Halliwell, 1991), the correlation among physical variables selects classical trajectories which are solutions of the cosmological field equations. This is realized if the solution of the Wheeler-DeWitt equation shows oscillating behaviors. Such a solution is called 'the Wave Function of the Universe'. It is possible to show that the Hartle criterion can be related to Noether's symmetries. In other words, it can be shown that if symmetries exist, the wave function of the universe presents oscillating behaviors (Capozziello et al., 2012) and then the presence of Noether's Symmetries allows the emergence of classically observable universes. Here, without claiming to offer a complete picture, we want to briefly sketch the use of Noether's Theorem in Cosmology. In Sect. 13.2, we discuss the so-called 'Minisuperscape Approach to Quantum Cosmology'. Noether's symmetries for Lagrangian and Hamiltonian dynamical systems are discussed in Sect. 13.3. Applications to specific cosmological models, derived from some theories gravity, are presented in Sect. 13.4. In Sect. 13.5, we discuss the results and draw some general conclusions.

13.2 Minisuperspaces in Cosmology

In order to discuss Noether's symmetries in Cosmology, we have to take into account the concept of *Minisuperspace* which is the finite dimensional configuration space of dynamical variables describing a cosmological model. In general, a minisuperspace is a restriction of the *Superspace*, the infinite-dimensional configuration space of a given theory of gravity, where the forms of the metric and matter fields are fixed by some symmetries, e.g. homogeneous, spherical, cylindrical, etc. The simplest minisuperspace models are constructed according to the Cosmological Principle by choosing homogeneous and isotropic metrics and matter fields. Other forms of minisuperpaces can be achieved by following the Bianchi classification of universes (Marmo et al., 1985). A simple minisuperspace can be provided assuming a homogeneous lapse function $N = N(t)$, the shift functions $N^i = 0$ set to zero and a 4-metric of the form

$$ds^2 = -N^2(t)dt^2 + h_{ij}(\mathbf{x}, t)dx^i dx^j \,. \tag{13.1}$$

where h_{ij} is 3-metric homogeneous and it is described by a finite number of functions of *t*, $q^\alpha(t)$, where $\alpha = 0, 1, 2 \cdots (n-1)$.[1] We can recast the Hilbert-Einstein action, where the Lagrangian is $\mathcal{L} = \sqrt{-g}\,[R - 2\Lambda]$, with *R* the Ricci curvature scalar and Λ the cosmological constant, as

$$\mathcal{S}[h_{ij}, N, N^i] = \frac{m_P^2}{16\pi} \int dt\; d^3x\; N\sqrt{h} \left[K_{ij}K^{ij} - K^2 + {}^{(3)}R - 2\Lambda\right] \,, \tag{13.2}$$

where K^{ij} is the extrinsic curvature, *K* its trace, ${}^{(3)}R$ the curvature and $\sqrt{h}$ the determinant of the spatial 3-manifold, respectively. In general, one gets[2]

$$\mathcal{S}[q^\alpha(t), N(t)] = \int_0^1 dt N \left[\frac{1}{2N^2} f_{\alpha\beta}(q)\dot{q}^\alpha \dot{q}^\beta - U(q)\right] \equiv \int \mathcal{L} dt \,, \tag{13.3}$$

assembling together kinetic and configuration terms (see Capozziello et al., 2012 for details). The last equation has the form of a relativistic point particle action where the particles moves on a *n*-dimensional curved space-time with a self-interaction potential. The variation with respect to q^{α} , gives the equations of motion

$$\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{q}^{\alpha}}{N}\right)+\frac{1}{N^{2}}\Gamma^{\alpha}_{\beta\gamma}\dot{q}^{\beta}\dot{q}^{\gamma}+f^{\alpha\beta}\frac{\partial U}{\partial q^{\beta}}=0\,, \tag{13.4}$$

where $\Gamma^{\alpha}_{\beta\gamma}$ are the Christoffel symbols derived from the metric $f_{\alpha\beta}$. Varying with respect to *N*, one gets

$$\frac{1}{2\,N^{2}}f_{\alpha\beta}\dot{q}^{\alpha}\dot{q}^{\beta}+U(q)=0\,, \tag{13.5}$$

that is a constraint equation. Eqs. (13.4) and (13.5) describe geodesic motion in minisuperspace with a forcing term. The general solution of (13.4), (13.5) requires $(2n-1)$ arbitrary parameters to be found. Equations (13.4) and (13.5) have to be equivalent, respectively, to the 00 and *ij* components of the Einstein field equations. This statement is not guaranteed since assuming a choice for the metric into the action and then taking variations to derive the minisuperspace field equations does not necessarily yield the same field equations. However, it holds in several cases, in particular, for Bianchi models. In order to find the Hamiltonian, canonical momenta have to be defined, that is

$$p_{\alpha}=\frac{\partial\mathcal{L}}{\partial\dot{q}^{\alpha}}=f_{\alpha\beta}\frac{\dot{q}^{\beta}}{N}\,, \tag{13.6}$$

and the canonical Hamiltonian is

$$\mathcal{H}_{c}=p_{\alpha}\dot{q}^{\alpha}-\mathcal{L}=N\left[\frac{1}{2}f^{\alpha\beta}p_{\alpha}p_{\beta}+U(q)\right]\equiv N\mathcal{H}\,, \tag{13.7}$$

where $f^{\alpha\beta}(q)$ is the inverse metric on minisuperspace. The action in Hamiltonian form is

$$\mathcal{S} = \int_0^1 dt \left[p_\alpha \dot{q}^\alpha - N\mathcal{H} \right] . \tag{13.8}$$

A Lagrange multiplier is the lapse function N and then the Hamiltonian constraint has to be

$$\mathcal{H}(q^\alpha, p_\alpha) = \frac{1}{2} f^{\alpha\beta} p_\alpha p_\beta + U(q) = 0 \, . \tag{13.9}$$

If we want to deal with Quantum Cosmology, the canonical quantization procedure requires a time-independent wave function $\Psi(q^\alpha)$ that has to be annihilated by the quantum operator corresponding to the classical constraint (13.9). In other words, we need cosmological models that have to be quantized in view of discussing the initial condition from which observable universes emerge. The procedure gives rises to the Wheeler-DeWitt equation,

$$\hat{\mathcal{H}}(q^\alpha, -i\frac{\partial}{\partial q^\alpha})\Psi(q^\alpha) = 0 \, , \tag{13.10}$$

where $\Psi(q^\alpha)$ is the *Wave Function of the Universe*, that is the function indicating whether or not cosmological observables are correlated.[3] Since the metric $f^{\alpha\beta}$ depends on q, there is a factor ordering issue in (13.10). This may be solved by requiring that the quantization procedure is covariant in minisuperspace, that is unchanged by field redefinitions of the 3-metric and matter fields, $q^\alpha \to \tilde{q}^\alpha(q^\alpha)$. This fact restricts the possible operator orderings to

$$\hat{\mathcal{H}} = -\frac{1}{2}\nabla^2 + \xi\mathcal{R} + U(q) \, , \tag{13.11}$$

where ∇^2 and $\mathcal{R}$ are the Laplacian and curvature of the minisuperspace metric $f_{\alpha\beta}$ and ξ is an arbitrary constant, respectively. Before concluding this section, an important issue has to be addressed. It is how to interpret the

probability measure in Quantum Cosmology. Since the Wheeler-DeWitt equation is similar to the Klein-Gordon equation we can define a conserved current, which satisfies $\nabla \cdot J = 0$, in this way

$$J = \frac{i}{2}\left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*\right) . \tag{13.12}$$

As in the case of the Klein-Gordon equation (and, in general, of hyperbolic equations), the probability derived from such a conserved current can be affected by negative probabilities. Due to this shortcoming, the correct measure to use should be

$$dP = \left|\Psi(q^\alpha)\right|^2 dV , \tag{13.13}$$

where *dV* is a volume element of minisuperspace (see Capozziello et al., 2012; Halliwell, 1991).

13.3 The Noether Symmetry Approach

As we said before, minisuperspaces are restrictions of the Superspace of geometrodynamics. They are finite-dimensional configuration spaces on which point-like Lagrangians can be defined. Cosmological models of physical interest can be defined on such minisuperspaces, such as Bianchi models. According to the above discussion, a crucial role is played by the conserved currents that allow to interpret the probability measure and then the physical quantities obtained in Quantum Cosmology. In this context, the search for general methods to achieve conserved quantities and symmetries becomes relevant. The so-called *Noether Symmetry Approach* (Capozziello et al., 1996; Bajardi & Capozziello, 2022), as we will show, can be extremely useful for this purpose. Let $\mathcal{L}$ be a Lagrangian

defined on the tangent space of configurations $T\mathcal{Q} \equiv \{q_i, \dot{q}_i\}$, the vector field X is

$$X = \alpha^i(q)\frac{\partial}{\partial q^i} + \dot{\alpha}^i(q)\frac{\partial}{\partial \dot{q}^i}\,, \tag{13.14}$$

where dot means derivative with respect to t, and

$$L_X\mathcal{L} = X\mathcal{L} = \alpha^i(q)\frac{\partial \mathcal{L}}{\partial q^i} + \dot{\alpha}^i(q)\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\,, \tag{13.15}$$

is the Lie derivative L_X of $\mathcal{L}$. The condition

$$L_X\mathcal{L} = 0\,, \tag{13.16}$$

implies that the phase flux is conserved along X: this means that a constant of motion exists for $\mathcal{L}$ and Noether's theorem holds. In fact, taking into account the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} = 0\,, \tag{13.17}$$

it is easy to show that

$$\frac{d}{dt}\left(\alpha^i\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) = L_X\mathcal{L}\,. \tag{13.18}$$

If Eq. (13.16) holds,

$$\Sigma_0 = \alpha^i\frac{\partial \mathcal{L}}{\partial \dot{q}^i} \tag{13.19}$$

is a constant of motion. Alternatively, using the Cartan one —form

$$\theta_{\mathcal{L}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\,dq^i \tag{13.20}$$

and defining the inner derivative

$$i_X\theta_{\mathcal{L}} =< \theta_{\mathcal{L}}, X > \ , \tag{13.21}$$

we get, as above,

$$i_X \theta_{\mathcal{L}} = \Sigma_0 \,, \tag{13.22}$$

if condition (13.16) holds. This representation is useful to identify cyclic variables. Using a point transformation on vector field (13.14), it is possible to get[4]

$$\tilde{X} = (i_X dQ^k) \frac{\partial}{\partial Q^k} + \left[\frac{d}{dt} (i_X dQ^k) \right] \frac{\partial}{\partial \dot{Q}^k} \,. \tag{13.23}$$

If X is a symmetry also $\tilde{X}$ has this property, then it is always possible to choose a coordinate transformation so that

$$i_X dQ^1 = 1 \,, \quad i_X dQ^i = 0 \,, \quad i \neq 1 \,, \tag{13.24}$$

and then

$$\tilde{X} = \frac{\partial}{\partial Q^1} \,, \qquad \frac{\partial \tilde{\mathcal{L}}}{\partial Q^1} = 0 \,. \tag{13.25}$$

It is evident that Q^1 is the cyclic coordinate and the dynamics can be reduced (Arnold, 1978). However, the change of coordinates is not unique and a suitable choice is always important. Furthermore, it is possible that more symmetries are found. In this case, more than one cyclic variable exists. A reduction procedure by cyclic coordinates can be implemented in three steps:

- We choose a symmetry and obtain new coordinates as above. After this first reduction, we get a new Lagrangian $\tilde{\mathcal{L}}$ with a cyclic coordinate.
- We search for new symmetries in this new space and apply the reduction technique until it is possible.
- The process stops if we select a pure kinetic Lagrangian where all coordinates are cyclic.

Going back to the viewpoint of Quantum Cosmology, any symmetry selects a constant conjugate momentum since, by the Euler-Lagrange equations, it is

$$\frac{\partial \tilde{\mathcal{L}}}{\partial Q^i} = 0 \Longleftrightarrow \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{Q}^i} = \Sigma_i \,. \tag{13.26}$$

Vice versa, the existence of a constant conjugate momentum means that a cyclic variable must exist. In other words, a Noether symmetry exists. Further remarks on the form of the Lagrangian $\mathcal{L}$ are necessary at this point. We shall take into account time–independent, non–degenerate Lagrangians $\mathcal{L} = \mathcal{L}(q^i, \dot{q}^j)$, i.e.,

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \,, \quad \det\, H_{ij} \equiv \det\, ||\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j}|| \neq 0 \,, \tag{13.27}$$

where H_{ij} is the Hessian. As in standard analytic mechanics, $\mathcal{L}$ can be set in the form

$$\mathcal{L} = T(q^i, \dot{q}^i) - V(q^i) \,, \tag{13.28}$$

where T is a positive–defined quadratic form in the $\dot{q}^j$ and $V(q^i)$ is a potential term. The energy function associated with $\mathcal{L}$ is

$$E_{\mathcal{L}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\, \dot{q}^i - \mathcal{L}(q^j, \dot{q}^j) \,, \tag{13.29}$$

and by the Legendre transformation

$$\mathcal{H} = \pi_j \dot{q}^j - \mathcal{L}(q^j, \dot{q}^j) \,, \qquad \pi_j = \frac{\partial \mathcal{L}}{\partial \dot{q}^j} \,, \tag{13.30}$$

we get the Hamiltonian function and the conjugate momenta. Considering again the symmetry, the condition (13.16) and the vector field X in Eq. (13.14) give a homogeneous polynomial of second degree in the velocities plus an inhomogeneous term in the q^j. Due to (13.16), such a polynomial has to be identically zero and then each coefficient must be independently zero. If n is the dimension of the configuration space, i.e., the dimension of the minisuperspace, we get $\{1 + n(n+1)/2\}$ partial

differential equations whose solutions assign the symmetry, as we shall see in the examples in the next section. Such a symmetry is over–determined and, if a solution exists, it is expressed in terms of integration constants. In the Hamiltonian formalism, we have

$$[\Sigma_j, \mathcal{H}] = 0\,, \quad 1 \leq j \leq m\,, \tag{13.31}$$

as it must be for conserved momenta and the Hamiltonian has to satisfy the relation

$$L_\Gamma \mathcal{H} = 0\,, \tag{13.32}$$

in order to obtain a Noether symmetry. The vector Γ is defined by Marmo et al. (1985):

$$\Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i}\,. \tag{13.33}$$

These considerations can be applied to the minisuperspace models of Quantum Cosmology and to the 'Wave Function of the Universe. By a straightforward canonical quantization procedure, we have

$$\pi_j \longrightarrow \hat{\pi}_j = -i\partial_j\,, \tag{13.34}$$

$$\mathcal{H} \longrightarrow \hat{\mathcal{H}}(q^j, -i\partial_{q^j})\,. \tag{13.35}$$

If $|\Psi>$ is a *state* of the system, i.e., the 'Wave Function of the Universe,' the dynamics is given by

$$\mathcal{H}|\Psi>= 0\,, \tag{13.36}$$

which is the Wheeler-DeWitt equation. If a Noether symmetry exists, the reduction procedure outlined above can be applied and then, from (13.26) and (13.30), we get

$$\begin{aligned}
\pi_1 \equiv \frac{\partial \mathcal{L}}{\partial \dot{Q}^1} = i_{X_1}\theta_{\mathcal{L}} = \quad & \Sigma_1 \,, \\
\pi_2 \equiv \frac{\partial \mathcal{L}}{\partial \dot{Q}^2} = i_{X_2}\theta_{\mathcal{L}} = \quad & \Sigma_2 \,, \\
\ldots \quad \ldots\ldots \,, &
\end{aligned} \tag{13.37}$$

depending on the number of Noether symmetries. After the quantization procedure, we have

$$\begin{aligned}
-i\partial_1|\Psi>=&\Sigma_1|\Psi> \,, \\
-i\partial_2|\Psi>=&\Sigma_2|\Psi> \,, \\
\ldots\ldots \,, &
\end{aligned} \tag{13.38}$$

which are nothing else but translations along the Q^j axes singled out by the corresponding symmetries. Equations (13.38) can be immediately integrated and, being Σ_j real constants, we obtain oscillatory behaviors for $|\Psi>$ in the directions of symmetries:

$$|\Psi>=\sum_{j=1}^{m} e^{i\Sigma_j Q^j}|\chi(Q^l)> \,, \quad m<l\leq n \,, \tag{13.39}$$

where *m* is the number of symmetries, *l* are the directions where symmetries do not exist, *n* is the total dimension of minisuperspace. Vice versa, dynamics given by (13.36) can be reduced by (13.38) if and only if it is possible to define constant conjugate momenta as in (13.37), that is oscillatory behaviors of a subset of solutions $|\Psi>$ exist, only if Noether symmetries exist for dynamics. The *m* symmetries give first integrals of motion and then the possibility to select classical trajectories. In one and two–dimensional minisuperspaces, the existence of a Noether symmetry allows the complete solution of the problem and to get the full semi-classical limit of Quantum Cosmology (Capozziello & Lambiase, 2000).

In conclusion, we can state that in the semi-classical limit of Quantum Cosmology, the reduction procedure of dynamics, connected to the existence of Noether symmetries, allows to select a subset of the solution of Wheeler-DeWitt equation where oscillatory behaviors are found. This fact, in the framework of the Hartle interpretative criterion of the 'Wave Function of the Universe', gives conserved momenta and trajectories which can be interpreted as classical cosmological solutions. Vice-versa, if a subset of the solution of Wheeler-DeWitt equation present oscillatory behaviors, due to Eq. (13.16), conserved momenta exist and Noether's symmetries are present. In other words, *Noether's symmetries select classical universes and then are directly related to the validity of the Hartle criterion*. It is worth noticing that 'classical universes' mean dynamical systems where variables are correlated. In other words, 'observable universes.' In what follows, we will show that such a statement holds for general classes of minisuperspaces and allows to select exact classical solutions, i.e., the presence of Noether symmetries is a selection criterion for observable universes. In the next section, we will shown how they work for Extended Theories of Gravity, as scalar tensor, $f(R)$ and $f(T)$ gravity.

13.4 Noether's Symmetries in Cosmology

Let us consider realizations of the above approach in minisuperspace cosmological models derived from Extended Theories of Gravity. Such theories of gravity have been recently invoked to address, for example, problems like dark energy and dark matter in cosmology and astrophysics (Capozziello & De Laurentis, 2011) As we have seen, the Hartle criterion is directly connected to the

presence of Noether symmetries since oscillatory behaviors means correlations among variables (see Nojiri & Odintsov, 2007; Capozziello & Francaviglia, 2008; Capozziello, De Laurentis & Faraoni, 2009; Capozziello & De Laurentis, 2011; Nojiri & Odintsov, 2011).

Specifically, the approach can be connected to the search for Lagrange multipliers. In fact, imposing Lagrange multipliers allow to modify the dynamics and select the form of effective potentials. By integrating the multipliers, solutions can be obtained. On the other hand, the Lagrange multipliers are constraints capable of reducing dynamics. Technically they are anholonomic constraints being time-dependent. They give rise to field equations which describe dynamics of the further degrees of freedom coming from Extended Theories of Gravity (Nojiri & Odintsov, 2007; Capozziello & Francaviglia, 2008; Capozziello et al., 2009; Capozziello & De Laurentis, 2011; Nojiri & Odintsov, 2011). This fact is extremely relevant to deal with new degrees of freedom under the standard of effective scalar fields (Capozziello et al., 2010). Below, we give minisuperspace examples and obtain exact cosmological solutions. In particular, we show that, by imposing Lagrange multipliers, a given minisuperspace model becomes canonical and Noether's symmetries, if exist, can be found out.

13.4.1 Scalar-Tensor Gravity

A general action for a nonminimally coupled theory of gravity assume the following form

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[F(\phi)R + \frac{1}{2}g^{\mu\nu}\phi_\mu\phi_\nu - V(\phi)\right], \qquad (13.40)$$

where $F(\phi)$ and $V(\phi)$ are respectively the coupling and the potential of a scalar field.[5] The cosmological point-like

Lagrangian for a Friedman-Lemaître-Robertson-Walker (FLRW) minisuperspace in terms of the scale factor a is

$$\mathcal{L} = 6a\dot{a}^2 F + 6a^2\dot{a}\dot{F} - 6kaF + a^3\left[\frac{\dot{\phi}}{2} - V\right] . \tag{13.41}$$

The configuration space of such a Lagrangian is $\mathcal{Q} \equiv \{a, \phi\}$, i.e., a bidimensional minisuperspace. A Noether symmetry exists if Eq. (13.16) holds. In this case, the Noether vector is

$$X = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial \phi} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{\phi}} , \tag{13.42}$$

where α, β depend on a, ϕ . This vector field acts on the $\mathcal{Q}$ minisuperspace. The system of partial differential equation given by (13.16) is

$$F(\phi)\left[\alpha + 2a\frac{\partial \alpha}{\partial a}\right] + aF'(\phi)\left[\beta + a\frac{\partial \beta}{\partial a}\right] = 0 , \tag{13.43}$$

$$3\alpha + 12F'(\phi)\frac{\partial \alpha}{\partial \phi} + 2a\frac{\partial \beta}{\partial \phi} = 0 , \tag{13.44}$$

$$a\beta F''(\phi) + \left[2\alpha + a\frac{\partial \alpha}{\partial a} + \frac{\partial \beta}{\partial \phi}\right] F'(\phi) + 2\frac{\partial \alpha}{\partial \phi}F(\phi) + \frac{a^2}{6}\frac{\partial \beta}{\partial a} = 0 , \tag{13.45}$$

$$[3\alpha V(\phi) + a\beta V'(\phi)]a^2 + 6k[\alpha F(\phi) + a\beta F'(\phi)] = 0 . \tag{13.46}$$

The prime indicates the derivative with respect to ϕ . The number of equations is 4 as it has to be, being $n = 2$ the $\mathcal{Q}$ -dimension. Several solutions exist for this system (Capozziello & de Ritis, 1993; Capozziello & de Ritis, 1994; Capozziello et al., 1995). They determine also the form of the model since the system (13.43)–(13.46) gives α, β , $F(\phi)$ and $V(\phi)$. For example, if the spatial curvature is $k = 0$, a solution is

$$\alpha = -\frac{2}{3}p(s)\beta_0 a^{s+1}\phi^{m(s)-1}\,, \quad \beta = \beta_0 a^s \phi^{m(s)}\,, \tag{13.47}$$

$$F(\phi) = D(s)\phi^2\,, \quad V(\phi) = \lambda\phi^{2p(s)}\,, \tag{13.48}$$

where

$$D(s) = \frac{(2\,s+3)^2}{48(s+1)(s+2)}\,, \tag{13.49}$$

$$p(s) = \frac{3(s+1)}{2\,s+3}\,, \tag{13.50}$$

$$m(s) = \frac{2\,s^2+6\,s+3}{2\,s+3}\,, \tag{13.51}$$

and s, λ are free parameters. The change of variables (13.24) gives

$$w = \sigma_0 a^3 \phi^{2p(s)}\,, \quad z = \frac{3}{\beta_0\chi(s)}a^{-s}\phi^{1-m(s)}\,, \tag{13.52}$$

where σ_0 is an integration constant and

$$\chi(s) = -\frac{6\,s}{2\,s+3}\,. \tag{13.53}$$

Lagrangian (13.41) becomes, for $k = 0$,

$$\mathcal{L} = \gamma(s) w^{s/3}\dot{z}\dot{w} - \lambda w\,, \tag{13.54}$$

where z is cyclic and

$$\gamma(s) = \frac{2\,s+3}{12\sigma_0^2(s+2)(s+1)}\,. \tag{13.55}$$

The conjugate momenta are

$$\pi_z = \frac{\partial\mathcal{L}}{\partial\dot{z}} = \gamma(s)w^{s/3}\dot{w}\,, \quad \pi_w = \frac{\partial\mathcal{L}}{\partial\dot{w}} = \gamma(s)w^{s/3}\dot{z}\,, \tag{13.56}$$

and the Hamiltonian is

$$\tilde{\mathcal{H}} = \frac{\pi_z \pi_w}{\gamma(s) w^{s/3}} + \lambda w \,. \tag{13.57}$$

The Noether symmetry is given by

$$\pi_z = \Sigma_0 \,. \tag{13.58}$$

Quantizing Eq. (13.56), we get

$$\pi \longrightarrow -i\partial_z \,, \quad \pi_w \longrightarrow -i\partial_w \,, \tag{13.59}$$

and then the Wheeler-DeWitt equation

$$[(i\partial_z)(i\partial_w) + \tilde{\lambda} w^{1+s/3}] |\Psi >= 0 \,, \tag{13.60}$$

where $\tilde{\lambda} = \gamma(s)\lambda$. The quantum version of constraint (13.58) is

$$-i\partial_z |\Psi >= \Sigma_0 |\Psi > \ , \tag{13.61}$$

so that dynamics results reduced. A straightforward integration of Eqs. (13.60) and (13.61) gives

$$|\Psi >= |\Omega(w) > |\chi(z) > \propto e^{i\Sigma_0 z} \, e^{-i\tilde{\lambda} w^{2+s/3}} \,, \tag{13.62}$$

which is an oscillating wave function and the Hartle criterion is recovered. In the semi–classical limit, we have two first integrals of motion: Σ_0, i.e., the equation for π_z) and $E_{\mathcal{L}} = 0$, i.e., the Hamiltonian (13.57) which becomes the equation for π_w. Classical trajectories in the configuration space $\tilde{\mathcal{Q}} \equiv \{w, z\}$ are immediately recovered

$$w(t) = [k_1 t + k_2]^{3/(s+3)} \,, \tag{13.63}$$

$$z(t) = [k_1 t + k_2]^{(s+6)/(s+3)} + z_0 \,, \tag{13.64}$$

then, going back to $\mathcal{Q} \equiv \{a, \phi\}$, we get the *classical* cosmological behaviour

$$a(t) = a_0 (t - t_0)^{l(s)} \,, \tag{13.65}$$

$$\phi(t) = \phi_0 (t - t_0)^{q(s)} \,, \tag{13.66}$$

where

$$l(s) = \frac{2\,s^2 + 9\,s + 6}{s(s+3)}\,, \quad q(s) = -\frac{2\,s+3}{s}\,, \tag{13.67}$$

which means that the Hartle criterion selects classical and observable universes. Depending on the value of s, we can get power–law or pole–like behaviors. If $0 < l(s) \leq 1$, the standard perfect fluid solutions of Friedman equations are recovered.

13.4.2 Extending the Hilbert-Einstein Action: The Case of *f*(*R*) Gravity

Similar results can be also obtained by extending the Hilbert-Einstein action. Let us consider the action action

$$\mathcal{S} = \int d^4x\sqrt{-g}\,f(R)\,, \tag{13.68}$$

where $f(R)$ is a generic function of the Ricci scalar (Capozziello & De Laurentis, 2011). Rewriting the action as a point-like FLRW one, we obtain

$$\mathcal{S} = \int dt\mathcal{L}(a, \dot{a}; R, \dot{R})\,, \tag{13.69}$$

where dot means derivative with respect to the cosmic time. The scale factor a and the Ricci scalar R are the canonical variables constituting the configuration space $\mathcal{Q} \equiv \{a, R\}$, that is the related minisuperspace. This position could seem arbitrary since R depends on $a, \dot{a}, \ddot{a}$, but it is generally used in canonical quantization (see Vilenkin, 1985; Schmidt, 2004; Capozziello et al., 1998). The definition of R in terms of $a, \dot{a}, \ddot{a}$ introduces a constraint which eliminates second and higher order derivatives in action (13.69), and yields to a system of second order differential equations in $\{a, R\}$. Action (13.69) can be written as

$$\mathcal{S} = 2\pi^2 \int dt \left\{ a^3 f(R) - \lambda \left[R + 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] \right\} , \quad (13.70)$$

where the Lagrange multiplier λ is derived by varying with respect to R. It is

$$\lambda = a^3 f'(R) . \quad (13.71)$$

Here prime means derivative with respect to R. To recover the analogy with previous scalar–tensor models, let us introduce the auxiliary field

$$p \equiv f'(R) , \quad (13.72)$$

so that the Lagrangian in (13.70) becomes

$$\mathcal{L} = 6a\dot{a}^2 p + 6a^2 \dot{a}\dot{p} - 6kap - a^3 W(p) , \quad (13.73)$$

which is of the same form of (13.41) a part the kinetic term. This is an Helmholtz–like Lagrangian (Magnano & Sokolowski, 1994) and a, p are independent fields. The potential $W(p)$ is defined as

$$W(p) = h(p)p - r(p) , \quad (13.74)$$

where

$$r(p) = \int f'(R) dR = \int p dR = f(R) , \quad h(p) = R , \quad (13.75)$$

such that $h = (f')^{-1}$ is the inverse function of f'. The configuration space is now $\mathcal{Q} \equiv \{a, p\}$ and p has the same role of the above ϕ. Condition (13.16) is now realized by the vector field

$$X = \alpha(a,p)\frac{\partial}{\partial a} + \beta(a,p)\frac{\partial}{\partial p} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{p}} , \quad (13.76)$$

and explicitly it gives the system

$$p \left[\alpha + 2a\frac{\partial \alpha}{\partial a} \right] + a \left[\beta + a\frac{\partial \beta}{\partial a} \right] = 0 , \quad (13.77)$$

$$a^2\frac{\partial\alpha}{\partial p}=0\,, \tag{13.78}$$

$$2\alpha+a\frac{\partial\alpha}{\partial a}+2p\frac{\partial\alpha}{\partial p}+a\frac{\partial\beta}{\partial p}=0\,, \tag{13.79}$$

$$6k[\alpha p+\beta a]+a^2[3\alpha W+a\beta\frac{\partial W}{\partial p}]=0\,. \tag{13.80}$$

The solution of this system, i.e., the existence of a Noether symmetry, gives α, β and $W(p)$. It is satisfied for

$$\alpha=\alpha(a)\,,\qquad \beta(a,p)=\beta_0 a^s p\,, \tag{13.81}$$

where s is a free parameter and β_0 is an integration constant. In particular,

$$s=0\longrightarrow\alpha(a)=-\frac{\beta_0}{3}\,a\,,\quad \beta(p)=\beta_0\,p\,, \tag{13.82}$$

$$W(p)=W_0\,p\,,\quad k=0\,,$$

$$s=-2\longrightarrow\alpha(a)=-\frac{\beta_0}{a}\,,\quad \beta(a,p)=\beta_0\,\frac{p}{a^2}\,, \tag{13.83}$$

$$W(p)=W_1 p^3\,,\quad \forall\ k\,,$$

where W_0 and W_1 are constants. Let us discuss separately the solutions (13.82) and (13.83).

13.4.2.1 The Case $s=0$

The induced change of variables $\mathcal{Q}\equiv\{a,p\}\longrightarrow\tilde{Q}\equiv\{w,z\}$ can be

$$w(a,p)=a^3p\,,\quad z(p)=\ln p\,. \tag{13.84}$$

Lagrangian (13.73) becomes

$$\tilde{\mathcal{L}}(w,\dot{w},\dot{z}) = \dot{z}\dot{w} - 2w\dot{z}^2 + \frac{\dot{w}^2}{w} - 3W_0 w\,. \tag{13.85}$$

and, obviously, z is the cyclic variable. The conjugate momenta are

$$\pi_z \equiv \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{z}} = \dot{w} - 4\dot{z} = \Sigma_0\,, \tag{13.86}$$

$$\pi_w \equiv \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{w}} = \dot{z} + 2\frac{\dot{w}}{w}\,. \tag{13.87}$$

and the Hamiltonian is

$$\mathcal{H}(w,\pi_w,\pi_z) = \pi_w\pi_z - \frac{\pi_z^2}{w} + 2w\pi_w^2 + 6W_0 w\,. \tag{13.88}$$

By canonical quantization, reduced dynamics is given by

$$\left[\partial_z^2 - 2w^2\partial_w^2 - w\partial_w\partial_z + 6W_0 w^2\right]|\Psi>=0\,, \tag{13.89}$$

$$-i\partial_z|\Psi>=\Sigma_0\,|\Psi>\ . \tag{13.90}$$

Clearly we are imposing a simple factor ordering in the Wheeler-DeWitt equation (13.89). Immediately, the wave function has an oscillatory factor, being

$$|\Psi>\sim e^{i\Sigma_0 z}|\chi(w)>\ . \tag{13.91}$$

The function $|\chi>$ satisfies the Bessel differential equation

$$\left[w^2\partial_w^2 + i\frac{\Sigma_0}{2}\,w\,\partial_w + \left(\frac{\Sigma_0^2}{2} - 3W_0 w^2\right)\right]\chi(w) = 0\,, \tag{13.92}$$

whose solutions are linear combinations of Bessel functions $Z_\nu(w)$

$$\chi(w) = w^{1/2 - i\Sigma_0/4} Z_\nu(\lambda w)\,, \tag{13.93}$$

where

$$\nu = \pm\frac{1}{4}\sqrt{4 - 9\Sigma_0^2 - i4\Sigma_0}\,, \quad \lambda = \pm 9\sqrt{\frac{W_0}{2}}\,. \tag{13.94}$$

The oscillatory regime for this component depends on the reality of ν and λ. The 'Wave Function of the Universe' descending from the Noether symmetry (13.82) is then

$$\Psi(z,w) \sim e^{i\Sigma_0[z-(1/4)\ln w]}\, w^{1/2} Z_\nu(\lambda w)\,. \tag{13.95}$$

For large w, the Bessel functions have an exponential behavior, in such a way that the wave function (13.95) can be written as

$$\Psi \sim e^{i[\Sigma_0 z - (\Sigma_0/4)\ln w \pm \lambda w]}\,. \tag{13.96}$$

Due to the oscillatory behaviour of Ψ, the Hartle criterion is immediately recovered. By identifying the exponential factor of (13.96) with the action S_0 (see Bajardi & Capozziello, 2022), we can recover the conserved momenta π_z, π_w and select classical trajectories. Going back to the old variables, we get the cosmological solutions

$$a(t) = a_0 e^{(\lambda/6)t}\, \exp\left\{-\frac{z_1}{3}\, e^{-(2\lambda/3)t}\right\}, \tag{13.97}$$

$$p(t) = p_0 e^{(\lambda/6)t}\, \exp\left\{z_1\, e^{-(2\lambda/3)t}\right\}, \tag{13.98}$$

where a_0, p_0 and z_1 are integration constants. It is clear that λ plays the role of a cosmological constant and inflationary behavior is asymptotically recovered.

13.4.2.2 The Case $s = -2$

The new variables adapted to the foliation are now

$$w(a,p) = ap\,, \qquad z(a) = a^2\,. \tag{13.99}$$

and Lagrangian (13.73) assumes the form

$$\tilde{\mathcal{L}}(w, \dot{w}, \dot{z}) = 3\dot{z}\dot{w} - 6kw - W_1 w^3\,, \tag{13.100}$$

The conjugate momenta are

$$\pi_z = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{z}} = 3\dot{w} = \Sigma_1 \,, \tag{13.101}$$

$$\pi_w = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{w}} = 3\dot{z} \,. \tag{13.102}$$

The Hamiltonian is given by

$$\mathcal{H}(w, \pi_w, \pi_z) = \frac{1}{3}\,\pi_z \pi_w + 6kw + W_1 w^3 \,. \tag{13.103}$$

Going over the same steps as above, the 'Wave Function of the Universe' is given by

$$\Psi(z, w) \sim e^{i[\Sigma_1 z + 9kw^2 + (3W_1/4)w^4]} \,, \tag{13.104}$$

and the classical cosmological solutions are

$$a(t) = \pm\sqrt{h(t)} \,, \qquad p(t) = \pm\frac{c_1 + (\Sigma_1/3)\,t}{\sqrt{h(t)}} \,, \tag{13.105}$$

where

$$h(t) = \left(\frac{W_1\Sigma_1^3}{36}\right) t^4 + \left(\frac{W_1 w_1 \Sigma_1}{6}\right) t^3 + \left(k\Sigma_1 + \frac{W_1 w_1^2 \Sigma_1}{2}\right) t^2 + \\ + w_1(6k + W_1 w_1^2)\, t + z_2 \,. \tag{13.106}$$

w_1, z_1 and z_2 are integration constants. Immediately we see that, for large t

$$a(t) \sim t^2 \,, \qquad p(t) \sim \frac{1}{t} \,. \tag{13.107}$$

which is a power-law inflationary behavior. An extensive discussion of Noether symmetries in $f(R)$ gravity is found in Capozziello and De Felice (2008), Paliathanasis et al. (2011), Bajardi and Capozziello (2022).

13.4.3 Teleparallel Gravity: The Case of $f(T)$ Gravity

In analogy to the $f(R)$ gravity, a new sort of Extended Gravity, the so-called $f(T)$ theory, has been recently proposed by Cai et al. (2016). It is an extended version of the teleparallel gravity originally proposed by Einstein (1928, 1930) and more recently by Hayashi and Shirafuji (1979). Teleparallelism uses as dynamical object a vierbein field $e_i(x^\mu)$, $i = 0, 1, 2, 3$, which is an orthonormal basis for the tangent space at each point x^μ of the manifold: $e_i.e_j = \eta_{ij}$, where $\eta_{ij} = diag(-1, 1, 1, 1)$. Each vector e_i can be described by its components e_i^μ, $\mu = 0, 1, 2, 3$ in a coordinate basis, i.e., $e_i = e_i^\mu \partial_\mu$. Notice that Latin indexes refer to the tangent space, while Greek indexes label coordinates on the manifold.

The metric tensor is obtained from the dual vierbein as $g_{\mu\nu}(x) = \eta_{ij} e_\mu^i(x) e_\nu^j(x)$. Furthermore the Weitzenböck connection is used, whose non-null torsion is

$$T^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\nu\mu} - \hat{\Gamma}^\lambda_{\mu\nu} = e_i^\lambda(\partial_\mu e_\nu^i - \partial_\nu e_\mu^i). \tag{13.108}$$

rather than the Levi-Civita connection which is used in General Relativity. This tensor encompasses all the information about the gravitational field. The Lagrangian is built by the torsion scalar derived from (13.108) and its dynamical equations for the vierbein imply the Einstein equations for the metric. The teleparallel Lagrangian is

$$T = S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}, \tag{13.109}$$

where

$$S_\rho{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}{}_\rho + \delta^\mu_\rho T^{\theta\nu}{}_\theta - \delta^\nu_\rho T^{\theta\mu}{}_\theta) \tag{13.110}$$

and $K^{\mu\nu}{}_\rho$ is the contorsion tensor

$$K^{\mu\nu}{}_\rho = -\frac{1}{2}(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}), \tag{13.111}$$

which equals the difference between the Weitzenböck and Levi-Civita connections. Thus, the general action assume

the following form

$$\mathcal{S} = \int d^4x e f(T), \tag{13.112}$$

where $e = det(e^i_\mu) = \sqrt{-g}$. If matter couples to the metric in the standard form then the variation of the action with respect to the vierbein leads to the equations (Bengochea & Ferraro, 2009)

$$e^{-1}\partial_\mu(eS_i{}^{\mu\nu})f'(T) - e^\lambda_i T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu}f'(T) + S_i{}^{\mu\nu}\partial_\mu(T)f''(T) + \frac{1}{4}e^\nu_i f(T) = e_i{}^\rho T_\rho{}^\nu, \tag{13.113}$$

where a prime denotes differentiation with respect to T, $S_i{}^{\mu\nu} = e_i{}^\rho S_\rho{}^{\mu\nu}$ and $T_{\mu\nu}$ is the matter energy-momentum tensor. In order to derive the cosmological equations in a FLRW metric, we need to infer, as above, a point-like Lagrangian from the action (13.112). As a consequence, the infinite number of degrees of freedom of the original field theory will be reduced to a finite number. Then we can write

$$e^i_\mu = diag(1, a(t), a(t), a(t)), \tag{13.114}$$

where $a(t)$ is the cosmological scale factor, and the action is

$$\mathcal{S} = \int \mathcal{L}(a, \dot{a}, T, \dot{T})dt \tag{13.115}$$

considering a and T as canonical variables, whereas $\mathcal{Q} = \{a, T\}$ is the configuration space, and $\mathcal{TQ} = \{a, \dot{a}, T, \dot{T}\}$ is the related tangent bundle on which $\mathcal{L}$ is defined. As above, one can use the method of Lagrange multipliers to set T as a constraint of the dynamics. Selecting a suitable Lagrange multiplier and integrating by parts, the Lagrangian $\mathcal{L}$ becomes canonical and then we have[6]:

$$\mathcal{S} = 2\pi^2 \int dt\, a^3 \left[f(T) - \lambda\left(T + 6\frac{\dot{a}^2}{a^2}\right) - \frac{\rho_{m0}}{a^3}\right], \tag{13.116}$$

where λ is a Lagrange mutiplier. The variation with respect to T, gives

$$\lambda = f'(T)\,. \tag{13.117}$$

Therefore, the action (13.116) can be rewritten as

$$\mathcal{S} = 2\pi^2 \int dt\, a^3 \left[f(T) - f'(T)\left(T + 6\frac{\dot{a}^2}{a^2}\right) - \frac{\rho_{m0}}{a^3} \right], \tag{13.118}$$

and then the point-like Lagrangian reads

$$\mathcal{L}(a, \dot{a}, T, \dot{T}) = a^3 \left[f(T) - f'(T)T \right] - 6f'(T)a\dot{a}^2 - \rho_{m0}\,. \tag{13.119}$$

Substituting Eq. (13.119) into the Euler-Lagrange equation, we obtain

$$a^3 f''(T)\left(T + 6\frac{\dot{a}^2}{a^2}\right) = 0\,, \tag{13.120}$$

$$f(T) - f'(T)T + 2f_T H^2 + 4\left[f'(T)\frac{\ddot{a}}{a} + H f''(T)\dot{T} \right] = 0\,. \tag{13.121}$$

If $f''(T) \neq 0$, from Eq. (13.120), it is easy to find that

$$T = -6\left(\frac{\dot{a}}{a}\right)^2 = -6H^2\,. \tag{13.122}$$

This is the Euler constraint for dynamics. Substituting Eq. (13.122) into Eq. (13.121) and using $\ddot{a}/a = H^2 + \dot{H}$, we get

$$48H^2 f''(T)\dot{H} - 4f'(T)\left(3H^2 + \dot{H}\right) - f(T) = 0\,. \tag{13.123}$$

By a Legendre transformation on Lagrangian (13.119), we obtain the Hamiltonian constraint

$$\mathcal{H}(a, \dot{a}, T, \dot{T}) = a^3 \left[-6f'(T)\frac{\dot{a}^2}{a^2} - f(T) + f'(T)T + \frac{\rho_{m0}}{a^3} \right]. \tag{13.124}$$

Considering the total energy $\mathcal{H} = 0$ (de Ritis et al., 1990; Capozziello & De Felice, 2008; Vakili, 2008) and using

Eq. (13.122), we get

$$12H^2 f'(T) + f(T) = \frac{\rho_{m0}}{a^3}\,, \tag{13.125}$$

In summary, the point-like Lagrangian (13.119) yields all the equations of motion (Wei et al., 2012). To obtain the Noether symmetries, we substitute Eq. (13.119) into Eq. (13.15) imposing $L_X\mathcal{L} = 0$ and using the relations $\dot{\alpha} = (\partial\alpha/\partial a)\,\dot{a} + (\partial\alpha/\partial T)\,\dot{T}$, $\dot{\beta} = (\partial\beta/\partial a)\,\dot{a} + (\partial\beta/\partial T)\,\dot{T}$, we obtain

$$\begin{aligned} &3\alpha a^2\left[f(T) - f'(T)T\right] - \beta a^3 f''(T)T + \\ &- 6\dot{a}^2\left[\alpha f'(T) + \beta a f''(T) + 2a f'(T)\frac{\partial\alpha}{\partial a}\right] - 12a\dot{a}\dot{T}\frac{\partial\alpha}{\partial T} = 0\,. \end{aligned} \tag{13.126}$$

As above, requiring the coefficients of $\dot{a}^2$, $\dot{T}^2$ and $\dot{a}\dot{T}$ in Eq. (13.126) to be zero, we find

$$a\frac{\partial\alpha}{\partial T} = 0\,, \tag{13.127}$$

$$\alpha f'(T) + \beta a f''(T) + 2a f'(T)\frac{\partial\alpha}{\partial a} = 0\,, \tag{13.128}$$

$$3\alpha a^2\left(f(T) - f'(T)T\right) - \beta a^3 f''(T)T = 0\,. \tag{13.129}$$

that is the *Noether condition* (Capozziello & De Felice, 2008; Wei et al., 2012; Bajardi & Capozziello, 2021; Basilakos et al., 2013). The corresponding constant of motion (*Noether's charge*), reads

$$Q_0 = -12\alpha f'(T)a\dot{a}\,, \tag{13.130}$$

where Q_0 is a constant. A solution of Eqs. (13.127), (13.128) and (13.129) exists if explicit forms of α and β are found. In this case, as above, a symmetry exists. Obviously, from Eq. (13.127), it is easy to see that α is independent of T, and hence it is a function of a only, i.e., $\alpha = \alpha(a)$. On the other hand, from Eq. (13.129), we have

$$\beta a f''(T)T = 3\alpha \left(f(T) - f'(T)T \right) . \tag{13.131}$$

Multiplying by T Eq. (13.128), and then substituting Eq. (13.131) into it, we obtain

$$f'(T)T \left(2a\frac{d\alpha}{da} - 2\alpha \right) + 3\alpha f(T) = 0 . \tag{13.132}$$

One can perform a separation of variables and recast Eq. (13.132) as

$$1 - \frac{a}{\alpha}\frac{d\alpha}{da} = \frac{3f(T)}{2f'(T)T} . \tag{13.133}$$

Since the left-hand side is a function of a only and the right-hand side is a function of T only, they must be equal to a constant for consistency. For convenience, we let this constant be $3/(2n)$, and then Eq. (13.133) can be separated into two ordinary differential equations, i.e.,

$$nf(T) = f'(T)T , \qquad 1 - \frac{a}{\alpha}\frac{d\alpha}{da} = \frac{3}{2n} . \tag{13.134}$$

It is easy to find the solutions of these two ordinary differential equations, namely

$$f(T) = \mu_0 T^n , \qquad \alpha(a) = \alpha_0 \, a^{1-3/(2n)} , \tag{13.135}$$

where μ_0 and α_0 are integral constants. Obviously, $f(T)$ and $\alpha(a)$ are both power-law functions. Substituting Eqs. (13.134) and (13.135) into Eq. (13.131), we find that

$$\beta(a, T) = -\frac{3\alpha_0}{n} \, a^{-3/(2n)} \, T . \tag{13.136}$$

Therefore a Noether symmetry exists. Finally, we find out the exact solution $a(t)$ for this family of $f(T)$. Substituting Eqs. (13.134), (13.135) and (13.122) into Eq. (13.130), we obtain an ordinary differential equation for $a(t)$, namely

$$a^{c_1} \, \dot{a} = c_2 , \tag{13.137}$$

where

$$c_1 = \frac{3}{2n} - 1\,, \qquad c_2 = \left[\frac{Q_0}{-12\alpha_0\mu_0 n(-6)^{n-1}}\right]^{1/(2n-1)}\,. \tag{13.138}$$

It is easy to find that the general solution for Eq. (13.137) is

$$a(t) = -(1+c_1)(c_3 - c_2 t)^{1/(1+c_1)} = (-1)^{1+2n/3}\cdot\frac{3}{2n}\,(c_2 t - c_3)^{2n/3} \tag{13.139}$$

where c_3 is another integration constant. Obviously, in the late time $|c_2 t| \gg |c_3|$ and the universe experiences a power-law expansion. Requiring $a(t=0) = 0$, it is easy to see that the integration constant c_3 is zero. Finally, we have a behavior of the form

$$a(t) \sim t^{2n/3} \tag{13.140}$$

which is clearly a Friedman behavior. Note that the condition $n > 0$ is required to ensure that the universe is expanding (Wei et al., 2012). Similar considerations can be developed in the case of non-metric gravity as discussed in (Bajardi and Capozziello, 2023; Bajardi et al., 2020).

13.5 Discussion and Conclusions

Noether's symmetries play a fundamental role in Quantum Cosmology. They allow to identify conserved quantities selecting oscillating behaviors of the 'Wave Function of the Universe.' The existence of such quantities is related to constant momenta and to the possibility to solve dynamical systems whose solutions represent observable universes. Peaked behaviors (i.e., oscillations) of the 'Wave Function of the Universe' mean correlations among variables and then the possibility to obtain observable cosmological models, according to the Hartle criterion. It states that observable universes are solutions of dynamics as soon as correlations among physical variables are identified. Here,

we searched for Noether symmetries in some minisuperspaces of Extended Theories of Gravity in order to find out exact solutions.

The method has been worked out for scalar-tensor gravity, $f(R)$ gravity and $f(T)$ gravity. The existence of symmetries fixes, in particular, the form of Lagrangians allowing the resolution of dynamics. The common feature of such dynamical systems is that Lagrange multipliers, related to symmetries, can be found out. Such multipliers allow to reduce dynamics and then to achieve exact cosmological solutions. Here we have taken into account some simple models but more physically interesting systems can be considered and compared to cosmological observations as discussed in (Capozziello et al., 2007). In conclusion, symmetries can discriminate between observable, viable models and unviable models. In other words, they are not only an aesthetic criterion but also a cognitive approach to state what a physical system is.

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Footnotes

1 See Capozziello et al., 2012, and references therein.

2 Here, we have assumed as signature $(-,+,+,+)$. The function $f_{\alpha\beta}(q)$ is the reduced De Witt metric, and the integration of *t* can range from 0 to 1 by shifting *t* and rescaling the lapse function.

3 Clearly, this is not the probability to select a given metric related with a given configuration of fields. If so, we would have fully solved the problem of Quantum Gravity.

4 We indicate the quantities as Lagrangians and vector fields with a tilde if the non—degenerate transformation

$$Q^i = Q^i(q)\,, \quad \dot{Q}^i(q) = \frac{\partial Q^i}{\partial q^j}\,\dot{q}^j\,,$$

is performed. However the Jacobian determinant $\mathcal{J} = \| \partial Q^i/\partial q^j \|$ has to be non–zero.

5 Using physical units $8\pi G = c = \hbar = 1$, the standard Einstein coupling is recovered for $F(\phi) = -1/2$.

6 See prvious works by Capozziello and De Felice (2008), Basilakos et al. (2011, 2013), Capozziello et al. (2007), Vakili (2008).